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CONSTRUCTIONS IN CONSTRUCTIVE MATHEMATICS AND LOGIC

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Geometry springs to mind as a part of mathematics that where constructions abound: the solution to a geometrical problem comprises paradigmatically analysis of what is given, a synthesis of what is sought, and a demonstration that the *construction* so found ("synthesized") satisfies what is required. As a victim of "New Maths" during the 1960's, my geometrical knowledge is too slender to treat competently of this subject and instead I consider another area where constructions are equally prominent, namely that of *Constructive Mathematics*.

On merely linguistic grounds one would expect *constructive mathematics* and logic somehow to involve constructions, and this also proves the case. Already in his earliest writings on the topic, when explaining the notion of proposition that serves in his formalized system of logic from a couple of years earlier, Arend Heyting held that a proof of a proposition is a mathematical construction that satisfies certain conditions and that can itself be considered mathematically. Drawing upon work by Kolmogoroff, who interpreted the constructive logical calculus in terms of tasks or problems and the mathematical proofs as their solutions, Heyting also, as is well-known, gave clauses for the logical connectives and quantifiers that have later been given a streamlined form that has come to be known as BHK. Heyting also knew of Gentzen's Natural Deduction formulations of formal systems and used them in exposition of mathematical intuitionism.

In my lecture I want to give an overview of how the logical community reacted to the notion of a construction as a proof of a proposition. Note here that this is a notion novel with Heyting: previously all demonstration had taken place at the level of theorems, but now proofs pertain also to entities at the level of propositions, that is, *contents* of theorems. Several of the foremost foundational researchers confronted the issue: Gödel, for instance, had a lifelong interest in the matter (1933 talk, Zilsel lecture, and the various formulations of the *Dialectica* interpretation), but never reached a satisfying position. Stephen Kleene (*recursive realizability*) and Georg Kreisel (Theory of Constructions at LMPS I at Stanford) also engaged intensively with the issues. Gödel and Kreisel thought of constructions being given in a universe of all constructions. In Gödel's words, this is a ”vast totality”. The proof-relation “construction c proves A” was thought of as a *decidable* propositional function over this universe of all constructions:

∀c (c proves A or c does not prove A).

Furthermore, Gödel and Kreisel were worried by what they considered the impredicativity of constructive implication:

a proof of A⊃B is a function f such that ∀p(p proves A ⊃ f(p) proves B)

Here we have quantification over *all proofs*; among these there may already be proofs of A that have been built from implications. Furthermore, the right-hand side already uses implication. Kreisel's attempted solution to this conundrum will be sketched. Kreisel verged on the edge of the abyss of paradox, whence his pupil Goodman found it necessary to retreat into the use of partial functions and to *stratify* his universe of constructions. Artemov's *Logic of Proofs* may be seen as a latter-day version of the Gödel-Kreisel idea.

The 1960's saw a great deal of formal and philosophical work, by among others Kreisel (ubiquitous), Tait (computability of typed lambda terms) and Myhill (intuitionistic analysis). After the appearance of Bishop's *Constructive Mathematics* in 1967 it gradually emerged that mathematical constructivism, and Brouwerian Intuitionism, in particular, are perhaps not well captured by the customary formal systems. Kreisel's pertinent question in his Bucharest lecture at LMPS IV in 1971 is still relevant today:

> Was the (logical) language of the current intuitionistic systems obtained by uncritical transfer from languages which were, tacitly, understood classically?

In 1969 William Howard formulated his influential *Formulae-as-types Notion of Construction*. Howard's work used a sequent calculus, but it was recast by Per Martin-Löf into final form as an isomorphism between terms of a *typed* lambda calculus and Natural Deduction derivations after the style of Gentzen and Prawitz. This way of proceeding does give a stratification of proofs (of propositions) by means of *typing* them. Gödel was in close contact with Howard at the time, as was Kreisel, but neither seems to have realized that the decidability of the "proof-relation" need not to be taken in the sense of a propositional function that is defined over the universe of all constructions.

Instead Howard's device of typing the proof-relation

term ("construction") is of type A

not only gives a convenient stratification of constructions, but it also yields the desired decidability, not as application of a decidable propositional function, but under the guise of a so called type-checking algorithm, that operates – in Wittgenstein's *Tractarian* words – by "mechanical calculation *am Symbol allein*". In the lecture this difference between the typed and untyped accounts will be spelled out in more detail.

Background reading: articles by myself and Scott Weinstein in *Journal of Philosophical Logic* **12**(1983:2)

Basic knowledge of recursive realizability, natural-deduction normalization, and Gödel's *Dialectica* interpretation will aid comprehension.