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**Arnol'd, the Jacobi identity, and orthocenters. (English summary)**
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There are two distinct traditions in mathematics. The first one believes in the unity of all existing mathematics and values very highly unexpected uses of results and techniques of one area of mathematics in another, preferably distant, one. Practitioners of this tradition who look at geometry tend to believe that the continuity of space is one of its essential attributes, and thus believe in an intimate connection between geometry and the real numbers (understood as a unique structure, insensitive to potential disturbances of its uniqueness coming from set theory), take advantage of the manifold benefits to be had from its topological or differential structure, or else assume that the geometry in question is built over special classes of fields, to unleash the powerful results of algebraic geometry.

The second, and oldest one, looks at geometry (and by extension, at any area of mathematics) as consisting of statements on the words of a language, that are not there to be verified in an algebraic or a continuous realm, but to be deduced from other such statements, the main aim of this undertaking being the deduction of statements from the weakest possible set of some special statements that are endowed with a certain hard-to-define quality of simplicity.

In the second tradition, the theorem stating that the altitudes of any trilateral are concurrent was deduced, in a generalized form, from very weak axiom systems between 1929 and 1979. Its proof can be found on pages 27–29 in [J. Hjelmslev, Mat.-Fys. Medd. K. Dan. Vidensk. Selsk. **8** (1929), no. 11; JFM 55.0957.02], in [H. Toepken, Deutsche Math. **5** (1941), 395–401; MR0003711 (2,259c)], on pages 57–58 of [F. Bachmann, *Aufbau der Geometrie aus dem Spiegelungsbegriff*, Die Grundlehren der mathematischen Wissenschaften, Bd. XCVI, Springer, Berlin, 1959; MR0107835 (21 #6557)] (and on page 305 of its second edition of 1973 [Springer, Berlin, 1973; MR0346643 (49 #11368)]), in [F. Bachmann, Canad. J. Math. **19** (1967), 895–903. (1 plate); MR0216360 (35 #7194)] (repeated on page 55 (Satz 2.27) of [F. Bachmann, *Ebene Spiegelungsgeometrie*, Bibliographisches Inst., Mannheim, 1989; MR1002943 (90h:51017)]), and on page 40 of [R. Lingenberg, *Metric planes and metric vector spaces*, John Wiley & Sons, New York, 1979; MR0536348 (80i:51018)]. In the variant of 1967, it reads as follows: If $G$ is a group, $S$ a set of involutory generators of $G$, such that $g^{-1}sg \in S$ for all $g \in G$ and $s \in S$—with elements of $S$ called 'lines', with involutory products of two lines called 'points' (the two lines whose product is involutory being called 'orthogonal' ($\perp$)), the incidence of point $ab$ and line $l$ meaning that $abl$ is involutory—from the hypotheses that orthogonal lines intersect in no more than one point, that the orthogonal line through any point to any line exists, and (in case the point is not equal to the line) is unique, and that three lines $a, b, c$ with a common orthogonal $g$ form a pencil (i.e. $abc$ is a line), then the altitude theorem in the following general form holds: If $abc \neq 1$, $a \perp u$, $b \perp v$, $c \perp w$, and $ubc$, $avc$, $abw$ are lines, then $uvw$ is a line. Thus, thinking of $a, b, c$ as the sides of a trilateral (that need not pairwise intersect), and of $u, v, w$ as the altitudes to the sides of the trilateral formed by $a, b, c$, one gets that the three altitudes form a pencil, and thus this form of the theorem includes not only the cases in which the sides of the trilateral do not intersect, but also those in which the altitudes do not intersect. While the planes defined by the axiom system of Bachmann's *Aufbau der Geometrie aus dem*

*Spiegelungsbegriff* are embeddable in Pappian projective planes, and so there is a direct connection with fields of characteristic $\neq 2$, the planes defined by the axioms of the 1967 version of the altitude theorem cannot, in general, be embedded in projective planes, so there is no connection with fields whatsoever, and thus no room for 2-by-2 matrices with entries in that field, nor for the Jacobi identity for the commutator of such matrices.

In the first tradition, the altitudes theorem in hyperbolic geometry has attracted the attention of V. I. Arnol′d [J. Geom. Phys. **53** (2005), no. 4, 421–427; MR2125401 (2005k:53142)], who finds that it is the Jacobi identity that "forces the heights of a triangle to cross at one point" (and in [V. I. Arnol′d, Uspekhi Mat. Nauk **59** (2004), no. 1(355), 25–44; MR2068841 (2005e:01018)] that the Jacobi identity for the ordinary vector product "expresses the altitude theorem" in Euclidean geometry). The present paper takes Arnol′d's papers as its starting point, and attempts to present an as elementary as possible version of Arnol′d's Jacobi identity proof of the altitudes theorem. To this end, the author employs in his proof only "the most simple form of the Jacobi identity, namely, the Jacobi identity for the commutator $[A, B] = AB - BA$ of 2-by-2 matrices". The hyperbolic proofs offered are carried out in the Beltrami-Klein model of hyperbolic geometry and in the model of hyperbolic geometry from Chapter V of [W. Fenchel, *Elementary geometry in hyperbolic space*, de Gruyter Stud. Math., 11, de Gruyter, Berlin, 1989; MR1004006 (91a:51009)]. The author also deduces the altitudes theorem directly from Chasles's theorem ("If $A, B, C$ are three different points in the (real) projective plane, and $a, b, c$ are the points polar to the lines $BC, CA, AB$, respectively, $a \neq A$, $b \neq B$, $c \neq C$, then the lines $aA, bB, cC$ are concurrent.") and presents sufficient conditions for the altitudes of a hyperbolic triangle to intersect. He also presents a proof in spherical geometry, as well as several proofs in Euclidean geometry (one of which is that in Appendix III, §1, Exercise 1 of [A. A. Kirillov, *Lectures on the orbit method*, Grad. Stud. Math., 64, Amer. Math. Soc., Providence, RI, 2004; MR2069175 (2005c:22001)]).

The real answer to the question "what actually 'forces' the altitudes theorem to hold?" had been provided by Bachmann in 1967, for he proved that, given the most trivial assumptions on $(G, S)$ ($S$, a set of involutory elements, generates $G$, $g^{-1}sg \in S$ for all $g \in G$ and $s \in S$, orthogonal lines intersect in no more than one point, and the orthogonal line through any point to any line exists, and (in case the point is not equal to the line) is unique), the altitudes theorem is equivalent to the three reflections theorem for lines with a common perpendicular (i.e. if $g$ is orthogonal to each of $a, b, c$, then $abc$ is a line).

That neither the author nor the referees of this or of Arnol′d's papers were aware that these matters had been long settled in a vastly more general context (the author mentions only one source for a "synthetic" proof in hyperbolic geometry (over Euclidean ordered fields), namely an undergraduate geometry textbook [R. Hartshorne, *Geometry: Euclid and beyond*, Undergrad. Texts Math., Springer, New York, 2000; MR1761093 (2001h:51001)], where it is left as an exercise) is not surprising, as the overwhelming majority of mathematicians work within the first tradition and very often ignore that the second tradition even existed in the 20th century, mostly believing that all matters of a foundational nature in geometry (which are assumed to be rather pedagogical in nature) were settled by Hilbert in 1899. The study of the foundations of geometry is probably the only mathematical discipline in which it is quite common to skip a whole century of research (the 20th, to be precise), work within the framework of the 19th century, and be sure of success! The reasons for ignoring the arguably most important book in metric geometry since Euclid's *Elements*, namely Bachmann's *Aufbau der Geometrie aus dem Spiegelungsbegriff*, are mostly related to the outright rejection of the second tradition's underlying philosophy: that geometry is not to be seen as the study of pre-existing, uniquely determined structures, but rather as the study of the effect assumptions have on statements, with the ultimate aim of finding minimal systems of assumptions for all theorems of greater interest. For readers of English only, language may have prevented the acquaintance with this monumental work (although the English of Lingenberg's book, which is very close in spirit to Bachmann's, has not garnered more attention). Readers of Russian have the benefit of a translation by R. I. Pimenov [F. Bachmann, *The construction of geometry on the basis of the symmetry concept* (Russian), translated from the German by R. I. Pimenov, Izdat. "Nauka", Moscow, 1969; MR0248597 (40 #1848)]. *Victor V. Pambuccian*

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### References


1. V. I. Arnol'd, On teaching mathematics (in Russian) *Uspekhi Mat. Nauk* **53** (1998) 229–234; English translation: *Russian Math. Surveys* **53** (1998) 229–236. MR1618209 (99k:00011)
2. V. I. Arnol'd, A. N. Kolmogorov and the natural science (in Russian), *Uspekhi Mat. Nauk* **59** (2004) 25–44; English translation: *Russian Math. Surveys* **59** (2004) 27–46. MR2068841 (2005e:01018)
3. V. I. Arnol'd, Lobachevsky triangle altitude theorem as the Jacobi identity in the Lie algebra of quadratic forms on symplectic plane, *J. of Geometry and Physics,* **53** (2005) 421–427; Russian version: *Matematicheskoe prosvestchenie, Ser. 3,* **9** (2005) 93–99. MR2125401 (2005k:53142)
4. M. Audin, *Geometry,* Springer, Berlin, 2002.
5. J. L. Coolidge, *The Elements of Non-Euclidean Geometry,* Clarendon Press, Oxford, 1909.
6. H. S. M. Coxeter, *Non-Euclidean Geometry,* 6th ed., Mathematical Association of America, Washington, DC, 1998; University of Toronto Press, Toronto, 1942. MR0006835 (4,50a)
7. H. S. M. Coxeter, *Introduction to Geometry,* John Wiley, New York, 1989; reprint of 2nd ed., John Wiley, New York, 1969; John Wiley, New York, 1961. MR0123930 (23 #A1251)
8. H. S. M. Coxeter, *Projective Geometry,* Springer, Berlin, 2003; Blaisdell, New York, 1964. MR1335229 (96c:51001)
9. W. Fenchel, *Elementary Geometry in Hyperbolic Space,* Walter de Gruyter, Berlin, 1989. MR1004006 (91a:51009)
10. R. Hartshorne, *Geometry: Euclid and Beyond,* Springer, Berlin, 2005; corrected 4th printing of 1st ed., 2000. MR1761093 (2001h:51001)
11. T. Jørgensen, Compact 3-manifolds of constant negative curvature fibering over the circle, *Ann. of Math.* **106** (1977) 61–72. doi:10.2307/1971158 MR0450546 (56 #8840)
12. A. A. Kirillov, *Lectures on the Orbit Method,* Graduate Studies in Mathematics, vol. 64, American Mathematical Society, Providence, RI, 2004. MR2069175 (2005c:22001)
13. A. Marden, *Outer Circles,* Cambridge University Press, Cambridge, 2007. MR2355387 (2008i:57001)
14. J. Milnor, On the Geometry of the Kepler Problem, *Amer. Math. Monthly* **90** (1983) 353–365. doi:10.2307/2975570 MR0707149 (84i:70008)
15. D. Pedoe, *Geometry: A Comprehensive Course,* Dover, Mineola, NY, 1988; Cambridge University Press, London, 1970. MR1017034 (90h:51001)
16. E. G. Rees, *Notes on Geometry,* Springer, Berlin, 2005; 6th printing of 1st ed., 1983. MR0681482 (84c:51001b)

17. M. B. Skopenkov, Theorem about the altitudes of a triangle and the Jacobi identity (in Russian), *Matematicheskoe Prosvestchenie, Ser. 3* **11** (2007) 79–89.
18. J. Stillwell, *Geometry of Surfaces,* Springer, Berlin, 1995; corrected 2nd reprint of 1st ed., 1992. MR1171453 (94b:53001)
19. A. Weil, *Basic Number Theory,* Springer, Berlin, 1995; Die Grundlehren der mathematischen Wissenschaften, vol. 144, Springer, Berlin, 1967. MR1344916 (96c:11002)

*Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.*