Pro and Contra Hilbert: Zermelo's Set Theories*

Volker Peckhaus
Universität Paderborn
Fakultät für Kulturwissenschaften – Philosophie
Warburger Str. 100, D – 33098 Paderborn
E-mail: peckhaus@hrz.upb.de

1 Introduction

On 27 July 1941 Ernst Zermelo celebrated his 70th birthday. On this occasion he received congratulations from Paul Bernays, his former student and collaborator, who was at that time Privatdozent at the Eidgenössische Technische Hochschule in Zurich. In his response, dated 1 October 1941, Zermelo wrote the following:

> I'm very glad about the fact that still some of my former colleagues and collaborators remember me, while I have lost several of my friends by death. One just becomes more and more lonely, is therefore the more grateful for any friendly remembrance. [...] Of course, I have no illusions anymore about the effects of my essential life's work as it concerns "foundations" and set theory. As far as my name is mentioned at all, this is always done *only* in respect to the "principle of choice" for which I *never* claimed priority.[1]

As evidence Zermelo referred to a recent congress on foundations at Zurich, most likely the congress "Les fondements et la méthode des sciences mathématiques" in December 1938 (cf. Gonseth *1941*), where none of his papers published after 1904, namely both notes in the *Annalen der Mathematik* of 1908 and both papers published in the *Fundamenta Mathematicae* in 1930 and 1935, had been mentioned, as he observed, whereas "the dubious merits of a Skolem or Gödel were spinned out pretty much." In this letter, Zermelo also remembered the convention of the Deutsche Mathematiker-Vereinigung

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[1] Zermelo to Bernays, dated Freiburg, 1 October 1941, Bernays Papers, ETH Library Zurich, Hs. 975.5259.

at Bad Elster almost ten years before in September 1931, where his lecture "had been excluded from discussion because of an intrigue of the Vienna school represented by [Hans] Hahn and [Kurt] Gödel. Since then," he continued, "I have lost any desire to lecture on foundations. This is obviously the fate of everyone who is not backed by a 'school' or clique."[2] Zermelo closed: "But maybe the time will come when my work will be rediscovered and read again." This last remark sounds strange for our ears, I'm sure for Bernays' ears as well. Zermelo was a legend already during his lifetime. Today and also at that time his name was connected to the big debate on the axiom of choice used by Zermelo in 1904 for the proof of the well-ordering theorem. He was responsible for the axiomatization of set theory presented in 1908, which helped to establish set theory as a widely accepted mathematical theory. Today Zermelo's name is omnipresent in set theory in acronyms like "ZF" or "ZFC".

But Zermelo was more than the founder of axiomatized set theory. With his application of set theory to the theory of chess, he became one of the founders of game theory, and with his application of the calculus of variations to the problem of the navigation of aircrafts he pioneered navigation theory. With his recurrence objection in kinetic gas theory he annoyed Ludwig Boltzmann, and with a translation of Homer's Odyssee he pleased even philology experts. In short: Zermelo was famous in his time. Why this resigned self-assessment?

This paper tries to give a partial answer to this question. It surveys the different stages of Ernst Zermelo's considerations on set theory and philosophy of mathematics. It focusses on ontological and semantical aspects and compares them with David Hilbert's conceptions. This development is seen in the context of Zermelo's biography and his unsuccessful fate in academia.

2 Biographical Sketch

Zermelo's development as an academic teacher and researcher can be called a "scientific career" only with irony.[3] He was born in Berlin on 27 July 1871. He studied mathematics, physics, and philosophy at Berlin, Halle and Freiburg, finally graduating in Berlin in 1894 with a dissertation on the calculus of variations supervised by Hermann Amandus Schwarz (Zermelo *1894*). His main working fields were applied mathematics and theoretical physics, an interest which he kept all his life long. From 1894 to 1897 he worked as an assistant to Max Planck at the Institute of Theoretical Physics in Berlin.

[2]On the controversy between Gödel and Zermelo cf. Moore *2002*, on the events in Bad Elster, *ibid.*, 57–61.

[3]On Zermelo's biography cf. Moore *1982*, Peckhaus *1990a*, *1990b*, ch. 4.

In 1897 he moved to Göttingen where he made his *Habilitation* in applied mathematics in 1899, and subsequently taught as a *Privatdozent* and eventually *Titularprofessor*, financed by grants, since 1908 by a remuneration for the first German lectureship for mathematical logic, and students' fees. Under the influence of David Hilbert he converted his main working field to set theory and the foundations of mathematics.

In 1910 he was appointed to a full professorship for mathematics at the University of Zurich. Already in Göttingen he had fallen ill with tuberculosis, a disease which made him finally incapable to fulfil his teaching duties. As result of this he was not tenured after six years, and had to retire in 1916. From then on he lived mainly from the pension he got from the Zurich education department.

In 1921 he moved back to Germany taking residence in the Black Forrest near Freiburg. There he was discovered by the Freiburg mathematicians Lothar Heffter and Alfred Loewy. They initiated Zermelo's appointment to a honorary professorship at the University of Freiburg (it was not connected to any revenues besides students' fees). Zermelo taught at Freiburg University until 1935 when he was dismissed due to a denunciation for not having presented "Hitler's salute" properly. He was rehabilitated after the War, but, ill and almost blind, he never taught again. Zermelo died on 21 May 1953 in Freiburg, survived by his wife Gertrud, who celebrated her 100th birthday in 2002.

This was in fact no career, of course. Only between 1910 and 1916 Zermelo got an official salary. Besides this period of six years his revenues consisted in grants, student's fees and finally for 37 years the pension he received from the Cantonal Government in Switzerland.

3 Two Periods of Research

Zermelo's research on set theory and the foundations of mathematics was concentrated in two periods: 1901 to 1910 and 1927 to 1935. They correspond to two specific periods of research on the foundations of mathematics by David Hilbert and his collaborators in Göttingen. In the first period Hilbert elaborated his early axiomatic program, and Zermelo's work is clearly along the lines proposed by Hilbert, it is *pro* Hilbert. The second period falls into the time when Hilbert developed his proof-theory which was, because of its finitistic character, flatly rejected by Zermelo. His work at that time was *contra* Hilbert.

3.1 The First Period

The two main topics of Hilbert's foundational considerations during the first period were modern axiomatics and attempts to decide Cantor's continuum hypothesis.

Usually the story of Hilbert's philosophy of mathematics is written starting with his seminal *Grundlagen der Geometrie* (Hilbert *1899*), not really a book on method, but the application of a method, the axiomatic method, to Euclidean geometry.[4] Nevertheless, with this book modern axiomatics was created. Hilbert proceeded from three imagined systems of things (points, straight lines, planes) which he called, using the Kantian term, "thought things". He then described their interrelations in a set of 20 axioms. In addition he investigated this set of axioms as an *object in itself*, proving its completeness, the independence of the axioms, and its consistency. The latter proof was done by reducing the consistency of the geometrical axioms to the presupposed consistency of arithmetic. Therefore, a complete consistency proof was in fact postponed, and, at the same time, a new task was set: to find a consistent set of axioms for arithmetic. Hilbert presented his ideas concerning the foundations of arithmetic in September 1899, at the annual meeting of the Deutsche Mathematiker-Vereinigung which took place at Munich. In his lecture "Über den Zahlbegriff" (*1900a*) he elaborated the foundations of arithmetic—in his opinion the basic discipline of mathematics—independently of set-theoretic considerations. Due to the sketchy character of this paper, Hilbert did not carry out the meta-axiomatic investigations on independence of the axioms, completeness of the system and its consistency. Concerning the "necessary task" to prove consistency, he asserted, "only a suitable modification of known methods of inference" (*1900a*, 184) was required. These optimistic words from September 1899 seem to indicate that Hilbert probably underestimated the enormity of the task in hand. He soon changed his views. In August 1900, less than one year later, he included the consistency proof for arithmetic as the second among his famous mathematical problems which he presented to the Second International Congress of Mathematicians at Paris (cf. Hilbert *1900b*). Three years later, on 27 October 1903, he again emphasized the distinguished rôle of the consistency proof in a lecture delivered before the Göttingen Mathematical Society on the foundations of arithmetic. Following the report, it was Hilbert's aim "to work out the 'axiomatic' standpoint clearly". On the rôle of consistency, he then coined the brief formula: "the principle of contradiction the pièce de résistance."[5]

[4]For the prehistory and development cf. Toepell *1999*.

[5]Cf. the report in *Jahresbericht der Deutschen Mathematiker-Vereinigung* **12** (1903), 592.

In Hilbert's opinion around the turn of the century, set theory was not the basic foundation of mathematics. Nevertheless, he was interested in set theory, as a mathematical theory. This becomes evident in his correspondence with Georg Cantor between 1897 and 1900.

The main topic of this exchange of letters were Cantor's problems with the assumption of the set of all cardinals. Already in the first of Cantor's letters to Hilbert, dated 26 September 1897 (Cantor *1991*, no. 156, 388–389), Cantor proves that the totality of alephs does not exist, i. e., that this totality is no well-defined, ready set (*fertige Menge*). If it is taken to be a ready set, a certain larger aleph would follow on this totality. So this new aleph would at the same time belong to the totality of all alephs, and not belong to it, because of being larger than all alephs (*ibid.*, 388).

Although this is a negative existence proof, the feature discussed was later called "Cantor's Paradox", i. e., the paradox of the greatest cardinal, or of the set of all cardinals. We have evidence that it was during his discussions with Cantor that Hilbert formulated a paradox of his own, later known in Göttingen as "Hilbert's Paradox" (cf. Peckhaus/Kahle *2002*), and first written down in Hilbert's unpublished lecture course *Logische Principien des mathematischen Denkens* presented to his Göttingen students in the summer-term of 1905.[6] Hilbert considered this paradox, resulting from the set-formation principles of union and self-mapping, as "purely mathematical" because he carefully avoided using any concept from transfinite arithmetic. It is this paradox Hilbert referred to in his letter to Gottlob Frege of 7 November 1903 after having received the second volume of Frege *Grundgesetze der Arithmetik* containing Freges admission that the logical system used there for the foundation of arithmetic had proved to be inconsistent. In this letter Hilbert referred to Frege's description of Russell's paradox in the postscript, and wrote that "this example" was already known in Göttingen. In a footnote he added "I believe Dr Zermelo discovered it three or four years ago after I had communicated my examples to him."[7] This quote gives evidence for a discourse between Hilbert and Zermelo on set theory that must have taken place already before the turn of the century. The quotation also indicates, however, that the revolutionary impact of the paradoxes was not seen in Göttingen before their effect on Frege's logic had become evident. Hilbert's early solution of Cantor's and his own paradox as presented in "Über den Zahlbegriff" (Hilbert *1900a*) and in the Paris problems lecture was simply to apply the axiomatic method to set theory. If the consistency proof for the axioms was successful, the existence of the totality of real numbers would have been shown at the same time, and the existence of the totality of all

[6] Hilbert *1905c*. For an analysis see Peckhaus *1990b*, 58–72.

[7] Frege *1980*, 51; German original Frege *1976*, 79–80.

powers or of all Cantorian alephs could be disproved (*1900a*, 184). Hilbert clearly saw the connections with the continuum problem which he listed in 1900 as the first of his "Mathematical Problems."

Besides Zermelo's involvement in the early discussion on the formation of unintended sets, what were his contributions in the framework of Hilbertian foundational research in this first period?

Already in the winter term 1900/01 he gave a course on set theory in Göttingen.[8] In 1902 he published a first paper on the addition of transfinite cardinals (Zermelo *1902*). In August 1904 he disproved Julius König's rejection of the continuum hypothesis in the discussions at the Third International Congress of Mathematicians in Heidelberg.[9] One month later he communicated to Hilbert in a letter that he was able to prove the well-ordering theorem which Hilbert had named "the key for proving the continuum hypothesis" in the Paris problems lecture. The proof was subsequently published in the *Mathematische Annalen* almost immediately (Zermelo *1904*), keeping the letter form. It evoked a storm of protest in the mathematical community, above all directed against Zermelo's use of the principle of choice (cf. Moore *1982*). He reacted by publishing a new proof in 1908 together with a rejection of the main criticisms which was in parts very polemical (Zermelo *1908a*). In the same year Zermelo published an axiom system of set theory according to Hilbert's model (Zermelo *1908b*), sharing Hilbert's opinion that a "deepening of foundations" of a mathematical discipline was necessary as soon as this discipline was questioned because of foundational problems. That this situation was given for set theory because of the paradoxes was well known in Göttingen, at least in 1908.

As mentioned earlier, Zermelo followed Hilbert's suggestions concerning the structure of axiomatic systems. Zermelo only mentions, however, the necessity of proving the consistency of his axioms. He didn't carry this proof out, although he had intended to add such proof as becomes evident from his correspondence with Hilbert. He stopped his research because Hilbert urged him to publish his results.[10] As we know today, a wise decision.

But Zermelo did not work exclusively in set theory! After the significance of the paradoxes had been understood in Göttingen, a new field of research was opened: logic. It then became evident that a consistency proof for arithmetic could not be found by slightly revising existing methods of inference,

[8] Cf. the notebook with the manuscript of this course (partially in shorthand) in the Zermelo papers, University Archives (UA), Freiburg i. Br., C 129/150. For a discussion cf. Moore *1982*, 155–156.

[9] For a report on this incident cf. Kowalewski *1950*, 198–203.

[10] On Zermelo's attempts to proof the consistency of his axiom system, cf. his letter to Hilbert, dated Arosa, 25 March 1907, Hilbert papers, Staats- und Universitätsbibliothek Göttingen, Cod. Ms D. Hilbert 447, fol. 5.

as Hilbert initially had assumed (Hilbert *1900a*, 184). It is, namely, pointless to rely on inferences based on a logic which had recently been proved to be inconsistent. First attempts of creating a new logic for his projects had been made by Hilbert himself in 1905, in the paper "Über die Grundlagen der Logik und der Arithmetik" (*1905a*), and elaborated in the lecture course on the logical principles of mathematical thought. Zermelo then took over the task of creating logical competence in Göttingen with his lecture course on mathematical logic in the summer term of 1908, the first one based on an official assignment.[11]

There are furthermore indications that in this period Zermelo followed Hilbert's idealistic attitude towards mathematical objects. As mentioned earlier, Hilbert regarded mathematical objects as "thought things", creations of the mind whose ontological status was simply left open. Existence of mathematical objects was seen as their consistent possibility within given theories. Hilbert's "ontology" is therefore without any realistic commitment. Hilbert style axiomatics of the pre-war era was epistemologically and ontologically neutral.

This attitude of keeping epistemological and ontological questions open was shared by Zermelo. He started the axiomatization of set theory in set theory itself as it was historically given. He then attempted "establishing the principles which are necessary to found this mathematical discipline" (Zermelo *1908a*, 261). The methodological tool he used was thus regressive analysis (cf. Peckhaus *2002*). The principles had to be restricted in such a way that all paradoxes could be avoided, but they had to be wide enough to retain everything of value in set theory. Zermelo mentioned the necessity of investigating the independence of the axioms, but concerning deeper philosophical considerations he remarked: "The further, more philosophical, question about the origin of these principles and the extent to which they are valid will not be discussed here" (Zermelo *1908a*, 262; transl. 200). However, Zermelo obviously didn't follow the ideology often connected with the modern axiomatic method according to which it helped to solve all philosophical problems of mathematics within mathematics itself, as claimed, e. g., by Paul Bernays (Bernays *1922*, 94). Zermelo, like Hilbert, simply avoided dealing with philosophical problems.

What Zermelo did, in fact, was to keep the philosophical impact on mathematics to a minimum. For the mathematician doing mathematics it is of no use to know whether the mathematical objects he is operating with have any real analogue, whether the signs he uses have a real world reference, or whether his statements are true in a referential sense. He is interested in the question whether his operations are possible, i. e., he is looking for a guarantee

[11] Zermelo *1908c*; cf. Peckhaus *1990b*, 106–116; *1990a*, *1992*, *1994*.

that his axioms do not imply any contradiction. Again the questions whether or not mathematical objects have real existence, whether or not mathematical theorems are true, are left open. It is simply *assumed* that the objects exist and that the theorems are true, *as if* the philosophical justification had already been given. The axioms are thus stipulated as hypotheses, and the axiom system on their base is a hypothetico-deductive systems. This position was introduced by Mario Pieri (cf. Marchisotto *1993*, 292), and taken over by Zermelo in his 1908 lecture course on mathematical logic.

Zermelo starts his logic course with considerations on the nature of mathematical judgements, especially the question whether arithmetical statements are analytic or synthetic. He opposes Frege, Peano and Russell on the "analytic side" with Poincaré on the "synthetic side", but refuses to decide on one of the factions. He prefers to hold a mediating position (*1908c*, 3). "We initially assume that synthetic and analytic judgements occur side by side in arithmetic and make it our business to isolate the analytical part." Zermelo now introduces a method which he calls "analytical reduction" which, as he says, goes back to Euclid (or, to be more exact, to Pappus of Alexandria), but was perfected in then recent times by Hilbert:

> This method consists in completely taking apart the proof of a theorem into syllogisms, and in completely anteposing all premisses used in the proof. One can now assert these premisses categorically, include them as hypotheses into the theorem. We can say, however: in general, mathematical statements are not yet analytical judgements, but we are able *to reduce them to analytical judgements* by hypothetical addition of synthetic premisses. The logically reduced mathematical theorems emerging in this way are analytical-hypothetical judgements and they form the logical skeleton of a mathematical theory.

Mathematical deductions are thus independent of the truth of their initial statements. It is not the task of a mathematician to determine the truth of the axioms.

3.2 The Second Period

The second period is characterized by the so-called foundational crisis between intuitionism and formalism released not by the two protagonists of these positions, L. E. J. Brouwer and David Hilbert, but set into being by Hermann Weyl's paper on the new foundational crisis of mathematics (Weyl *1921*), thus constituting a self-fulfilling prophecy. Hilbert's answer to Brouwer's criticism of the use of the law of the excluded middle in infinite domains was the creation of meta-mathematics or proof theory (Hilbert *1922*). Proof theory is no proper mathematics, but the investigation of methods used in proofs, and aiming especially at a finite justification of means which can be used to deal with infinite domains.

In this second period we find Zermelo in a less isolated situation than in the years after his dismissal from the Zurich professorship. After having been called to the honorary professorship in Freiburg, he took part in the mathematical life in Freiburg, and started to publish again, supported and motivated by a funding from the Emergency Community of German Science, the precursor of the German Research Foundations (DFG), for a project entitled "Nature and Foundations of Pure and Applied Mathematics, and the Meaning of the Infinite in Mathematics" which helped him to improve his income between 1929 and 1931.[12] Zermelo's investigations resulted in three fundamental papers on cumulative hierarchies of sets and infinitary languages (Zermelo *1930*, *1932a*, *1935*). A fresh impetus to his research was furthermore given by a lecture tour of four months through Poland where he was able to discuss his new ideas on the foundations of set theory. In a report for the Emergency Community from December 1930 he motivated his rather late entering the debate on the foundational crisis in mathematics, the struggle between Hilbert and Brouwer. He stressed that he had started research on foundations 30 years before under the influence of David Hilbert, "to whom I owe the most in my scientific development," and that his contributions found a preliminary conclusion with his axiomatization of set theory which essentially remained definitive in axiomatic research in set theory. "In the meantime," he continued,

> the question of "foundations" got going again by the somewhat noisy appearance of the "intuitionists" who proclaimed a "foundational crisis" in mathematics in impetuous pamphlets and declared war on so to say the whole of modern science—without being able to put anything better at its place. One of its most officious adepts decreed "A set theory as a special mathematical discipline will not exist any more" while at the same time the new text books of set theory run to leaf. This state of affairs prompted me at that time to redirect my research activities to foundational problems, after having been almost alienated from scientific production by a lengthy illness and mental isolation abroad. Without becoming a party liner in this proclaimed dispute between "intuitionism" and "formalism"—I think that this alternative is an application of the "tertium non datur" which is logically inadmissable, anyway—I believed to be able to contribute to a clarification of the relevant questions.[13]

No doubt, the most important of these contributions was Zermelo's paper

[12] Cf. UA Freiburg, C 129/40. Zermelo's report for the Emergency Community of German Science was published by Gregory H. Moore (Moore *1980*, 130–134).

[13] UA Freiburg C 129/140; Moore *1980*, 131.

"On Boundary Numbers and Domains of Sets: New Investigations in the Foundations of Set Theory", published in 1930, which he described in a letter to the Warsaw mathematician Władisław Sierpiński[14] as dealing with the foundations of set theory and promising

> to give a satisfactory clarification of the so-called antinomies [...]. It concerns the investigation of such "domains of sets", in which the general axioms of set theory are satisfied, and the systematic development of their, essentially different (not isomorphic) "models" which can serve as their representations.

In this pioneering work Zermelo anticipated the recent modal structuralism inspired by Hilary Putnam (*1967*) and elaborated by Geoffrey Hellman (*1989*). Its essential result is, as Hellman put it (*1989*, 55–56): "Set theory should be seen, not as the theory of a unique, all-embracing, but instead as a theory of an endless infinity of intimately related structures."

Zermelo suggests an axiomatization that he himself calls "*ZF* system" for "Zermelo-Fraenkel system",[15] and which he enlarges to the "supplemented *ZF* system" ZF' by adding the axiom of foundation.

The paper can be regarded as a delayed, and in the beginning unconscious, rejoinder to Thoralf Skolem's criticism of his first axiomatization of set theory (Skolem *1923*).[16] This criticism concerned above all the axiom of separation according to which for every definite class statement $\mathfrak{E}(x)$ for a set M there is a subset N of M which contains those elements x of M for which $\mathfrak{E}(x)$ is true (Zermelo *1908b*). The notion of definiteness used in this formulation became controversial subsequently because it was introduced by Zermelo in an informal way. According to him a statement is called "definite" if one can decide about its validity or invalidity "without arbitrariness" with the help of "generally valid logical laws" (Zermelo *1908b*, 262). Skolem (Skolem *1923*) sharpened Zermelo's vague reference "generally valid logical laws," by presenting Zermelo's set theory in a first-order language and showing that the Löwenheim-Skolem theorem also holds for Zermelo's set theory. This theorem says that each finite or countably infinite set of statements of the first-order logic which has a model, has a countable model.

Zermelo had refined his notion of definiteness in 1929 without knowing of Skolem's criticism (Zermelo *1929*), but again he was countered at once

[14]Zermelo to Sierpiński, dated Freiburg, 26 March 1930, UA Freiburg, C 129/100. On this paper cf. Michael Hallett's introduction (Hallett *1996*) to its English translation, and furthermore Lavine *1994*, 134–141; Moore *1980*, 120–130; Moore *1987*, 125–128; Kanamori *1996*, 26–29; and Ebbinghaus *2002*.

[15]Today this system is "ZFC" (with "C" for "[axiom of] choice") because the axiom of choice is presupposed by Zermelo as a "general logical principle" (Zermelo *1930*, 31).

[16]On the relation between Skolem and Zermelo, especially on the chronology of Zermelo's reactions to "Skolemism" see van Dalen/Ebbinghaus *2000*, Ebbinghaus *2002*.

(Skolem *1930*).

Zermelo's paper on boundary numbers belongs to the context of Zermelo's fight against the "Skolemism",[17] i. e., any kind of finitism. It is interesting to see that Zermelo opposed almost all foundational positions at that time. He particularly didn't follow Hilbert's move towards metamathematics. In metamathematics Hilbert left his ontological and epistemological neutrality and proposed a constructive or operative way of founding mathematics which comes close to Brouwer's intuitionism in its restriction to finite operations. Zermelo, however, rejected any finitistic approach to mathematics as expression of a "Skolemism" in set theory. Alternatively he kept his idealistic approach, attempting to justify his infinite hierarchies of sets with the help of what he called a "logic of the infinite" (cf. Zermelo *1932a,b*), with the help of which he wanted to counter the "shortcomings of any 'finitistic' proof theory" (Zermelo *1932a*, 87). With this he became one of the early precursors of modern infinitary logic (cf. Moore *1997*).

4 Conclusion

In concluding, let me come back to my initial question about the reasons for Zermelo's resignation as expressed in his letter to Bernays of October 1941. It was the loneliness of the living legend increased by the fact that in foundations he was out of the professional mainstream for many years, due to his persistent illness, his scientific isolation, and his continuing active interest in applied mathematics. He did not participate especially during the period of heated debates on logic and the foundations of mathematics after the First World War. Zermelo didn't succeed in turning the prevailing debate characterized by its finitistic metamathematical spirit into the direction of his infinitistic ideas. As the development from the 1950s shows, they were ingenious for their time, but came 20 years too early.

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[17] This term is used in Zermelo *1932a*, 85.

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