LECTURES ON ALGEBRAIC VARIETIES OVER $\mathbb{F}_1$

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Contents

1. Introduction

These notes are issued from a series of lectures given at the Seventh Annual Spring Institute on Noncommutative Geometry and Operator algebras, 2009, in Vanderbildt University. They present a summary of the author's article [8], with a few modifications. These are made in order to take into account corrections and improvements due to A.Connes and C.Consani [1, 2]. The notion of *affine gadget over* $\mathbb{F}_1$ introduced below (Def. 3.2) lies somewhere in between the notion of "truc" in [8],3.1., Def.1 (see however i) in loc. cit.) and A.Connes and C.Consani's notion of "gadget over $\mathbb{F}_1$" [1]. We also added a discussion of the article of R.Steinberg [9] on the analogy between symmetric groups and general linear groups over finite fields.

I thank P.Cartier, A.Connes, C.Consani, J. Lopez Pena and O.Lorscheid for helpful discussions.

2. Preliminaries

2.1. **An analogy.** There is an analogy between the symmetric group Σ_n on n letters and the general linear group $GL(n, \mathbb{F}_q)$, where $q = p^k$ for a prime p. One of the first to write about this analogy was R.Steinberg in 1951 [9]. He used it to get a result in representation theory.This goes as follows.

For all $r \in \mathbb{N}$, define

$$[r] = q^{r-1} + q^{r-2} + \cdots + q + 1 = \frac{q^r - 1}{q - 1} \text{ and}$$

$$\{r\} = \prod_{i=1}^{r} [i].$$

Let $n \geq 1$ and $G = GL(n, \mathbb{F}_q)$. Let $\nu = (\nu_1, \ldots, \nu_n)$ be a partition of n, i.e.

$$n = \sum_{i=1}^{n} \nu_i \text{ where } 0 \leq \nu_1 \leq \nu_2 \leq \cdots \leq \nu_n.$$

Write every element of G as an n by n matrix of blocks of size $\nu_i \times \nu_j, 1 \leq i, j \leq n$. Consider the (parabolic) subgroup of upper triangular such matrices

$$G(\nu) = \left\{ g = \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix} \right\} \subset G\,.$$

One checks that

$$\#G/G(\nu) = \frac{\{n\}}{\prod_{i=1}^{n} \{\nu_i\}}.$$

Let

$$C(\nu) = \mathrm{Ind}_{G(\nu)}^{G} \mathbf{1} = \mathbb{C}[G/G(\nu)]$$

be the induced representation of the trivial representation of $G(\nu)$.

Theorem 2.1. *Let ν be a partition of n and $\lambda_i = \nu_i + i - 1$ for all $i \geq 1$. The virtual representation*

$$\Gamma(\nu) = \sum_{\kappa} \mathrm{sgn}(\kappa_1, \ldots, \kappa_n) C(\lambda_1 - \kappa_1, \ldots, \lambda_n - \kappa_n)$$

is an irreducible representation of G (i.e. its character is the character of an irreducible representation), when $\kappa = (\kappa_1, \ldots, \kappa_n)$ runs over all $n!$ permutations of $0, 1, \ldots, n-1$, with the convention that if $\lambda_i - \kappa_i < 0$ for some i, then $C(\lambda_1 - \kappa_1, \ldots, \lambda_n - \kappa_n) = 0$. Moreover, if $\Gamma(\mu) = \Gamma(\nu)$, then $\mu = \nu$.

To prove this result we consider the symmetric group $H = \Sigma_n$ and its subgroup

$$H(\nu) = \Sigma_{\nu_1} \times \cdots \times \Sigma_{\nu_n}.$$

Then

$$\#H/H(\nu) = \frac{n!}{\prod_{i=1}^{n} \nu_i!}.$$

Set

$$D(\nu) = \operatorname{Ind}_{H(\nu)}^{H} \mathbf{1} = \mathbb{C}[H/H(\nu)]$$

and consider the virtual representation

$$\Delta(\nu) = \sum_{\kappa} \operatorname{sgn}(\kappa_1, \ldots, \kappa_n) D(\lambda_1 - \kappa_1, \ldots, \lambda_n - \kappa_n).$$

Theorem 2.2 (Frobenius, 1898)**.** *$\Delta(\nu)$ is an irreducible representation. Moreover, $\Delta(\mu) = \Delta(\nu)$ implies $\mu = \nu$.*

The proof of 2.1 follows from this theorem and the following lemma:

Lemma 2.3. *Let $x \mapsto \psi(\nu, x)$ be the character of $C(\nu)$ and $x \mapsto \varphi(\nu, x)$ the character of $D(\nu)$. Then, for all μ, ν, we have*

$$\frac{1}{\#G} \sum_{x \in G} \psi(\nu, x)\psi(\mu, x) = \frac{1}{\#H} \sum_{x \in H} \varphi(\nu, x)\varphi(\mu, x)$$

Proof. The left hand side (resp. the right hand side) of this equality is the number of double cosets of G (resp.H) modulo $G(\mu)$ and $G(\nu)$ (resp. $H(\mu)$ and $H(\nu)$). We have an inclusion $H \subset G$ such that $H(\mu) = G(\mu) \cap H$, and the Bruhat decomposition implies that the map

$$H(\mu)\backslash H/H(\nu) \to G(\mu)\backslash G/G(\nu)$$

is a bijection. □

Let $\chi(\nu, x)$ be the character of $C(\nu)$. From the lemma and Frobenius' theorem we deduce that

$$\frac{1}{\#G} \sum_{x \in G} \chi(\nu, x)\overline{\chi(\mu, x)} = \delta_{\mu,\nu},$$

and, to get 2.1, it remains to check that $\chi(\nu, 1) > 0$.

2.2. **The field $\mathbb{F}_1$.** In [10] Tits noticed that the analogy above extends to an analogy between the group $G(\mathbb{F}_q)$ of points in $\mathbb{F}_q$ of a Chevalley group scheme G and its Weyl group W. He had the idea that there should exist a "field of characteristic one" $\mathbb{F}_1$ such that

$$W = G(\mathbb{F}_1)\,.$$

He showed furthermore that, when q goes to 1, the finite geometry attached to $G(\mathbb{F}_q)$ becomes the finite geometry of the Coxeter group W.

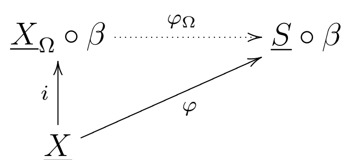
Thirty five years later, Smirnov [7], and then Kapranov and Manin, wrote about $\mathbb{F}_1$, viewed as the missing ground field over which number rings are defined. Since then several people studied $\mathbb{F}_1$ and tried to define algebraic geometry over it. Today, there are at least seven different definitions of such a geometry, and a few studies comparing them.

3. Affine varieties over $\mathbb{F}_1$

3.1. **Schemes as functors.** We shall propose a definition for varieties over $\mathbb{F}_1$ based on three remarks. The first one is that schemes can be defined as covariant functors from rings to sets (satisfying some extra properties, see [3]).

The second remark is that extension of scalars can be defined in terms of functors. Namely, let k be a field, and let Ω be a k-algebra. If X is a variety over k, we denote by $X_\Omega = X \otimes_k \Omega$ $(= X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(\Omega))$ its extension of scalars from k to Ω. Let $\underline{X}$ be the functor from k-algebras to sets defined by X and $\underline{X}_\Omega$ the functor from Ω-algebras to sets defined by X_Ω. Let β be the functor ${}_k\mathsf{Alg} \to {}_\Omega\mathsf{Alg}$ given by $R \mapsto R \otimes_k \Omega$.

Proposition 3.1.
(1) *There is a natural transformation* $i \colon \underline{X} \to \underline{X}_\Omega \circ \beta$ *of functors* ${}_k\mathsf{Alg} \to \mathsf{Set}$. *For any k-algebra R the map* $\underline{X}(R) \to \underline{X}_\Omega(R_\Omega)$ *is injective.*
(2) *For any scheme S over Ω and any natural transformation* $\varphi \colon \underline{X} \to \underline{S} \circ \beta$, *there exists a unique algebraic morphism* $\varphi_\Omega \colon X_\Omega \to S$ *such that* $\varphi = \varphi_\Omega \circ i$. *In other words the following diagram is commutative:*



We deduce from this proposition that, if X is a variety over $\mathbb{F}_1$,
(1) X should determine a covariant functor $\underline{X}$ from $\mathbb{F}_1$-algebras to Set;
(2) X should define a variety $X \otimes_{\mathbb{F}_1} \mathbb{Z}$ over $\mathbb{Z}$ by some universal property similar to the one in the proposition above (with $k = \mathbb{F}_1$ and $\Omega = \mathbb{Z}$).

3.2. **A definition.** A third remark is that we know what should play the role of finite extensions of $\mathbb{F}_1$. According to both Kapranov-Smirnov [4] and Kurokawa-Ochiai-Watanabe [5], the category of finite extensions of $\mathbb{F}_1$ is Ab_f, the category of finite abelian groups. If $D \in \mathsf{Ab}_f$, we define the extension of scalars of D from $\mathbb{F}_1$ to $\mathbb{Z}$ as the group-algebra $D \otimes_{\mathbb{F}_1} \mathbb{Z} = \mathbb{Z}[D]$. For example, $\mathbb{F}_{1^n} = \mathbb{Z}/n$, and

$$\mathbb{F}_{1^n} \otimes_{\mathbb{F}_1} \mathbb{Z} = \mathbb{Z}[T]/(T^n - 1)\,.$$

We now make the following

Definition 3.2. An *affine gadget over* $\mathbb{F}_1$ is a triple $X = (\underline{X}, \mathcal{A}_X, e_X)$ consisting of
(1) a covariant functor $\underline{X} \colon \mathsf{Ab}_f \to \mathsf{Set}$,
(2) a $\mathbb{C}$-algebra $\mathcal{A}_X$, and
(3) a natural transformation $e_X \colon \underline{X} \Rightarrow \mathrm{Hom}(\mathcal{A}_X, \mathbb{C}[-])$.

In other words, if $D \in \mathsf{Ab}_f$ and $P \in \underline{X}(D)$, we get a morphism of complex algebras $\mathcal{A}_X \to \mathbb{C}[D]$, that we write $e_X(P)(f) = f(P) \in \mathbb{C}[D]$, the *evaluation* of $f \in \mathcal{A}_X$ at the point P.

Example 3.3. Assume V is an affine algebraic variety over $\mathbb{Z}$. Then we can define an affine gadget $X = \mathcal{G}(V)$ as follows:

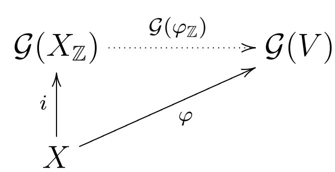
(1) $\underline{X}(D) = V(\mathbb{Z}[D])$,
(2) $\mathcal{A}_X = \Gamma(V_{\mathbb{C}}, \mathcal{O})$, and
(3) given $P \in V(\mathbb{Z}[D]) \subset V(\mathbb{C}[D])$ and $f \in \mathcal{A}_X$, then $f(P) \in \mathbb{C}[D]$ is the usual evaluation of the function f at P.

Definition 3.4. A *morphism* of affine gadgets $\phi\colon X \to Y$ consists of
(1) a natural transformation $\underline{\phi}\colon \underline{X} \to \underline{Y}$, and
(2) a morphism of algebras $\phi^*\colon \mathcal{A}_Y \to \mathcal{A}_X$,
which are compatible with evaluations, i.e. if $P \in \underline{X}(D)$ and $f \in \mathcal{A}_Y$, then $f(\underline{\phi}(P)) = (\phi^*(f))(P)$.

Definition 3.5. An *immersion* is a morphism $(\underline{\phi}, \phi^*)$ such that both $\underline{\phi}$ and ϕ^* are injective.

We can now define affine varieties over $\mathbb{F}_1$ as a special type of affine gadgets:

Definition 3.6. An *affine variety over* $\mathbb{F}_1$ is an affine gadget $X = (\underline{X}, \mathcal{A}_X, e_X)$ over $\mathbb{F}_1$ such that
(1) for any $D \in \mathsf{Ab}_f$, the set $\underline{X}(D)$ is finite;
(2) there exists an affine variety $X_{\mathbb{Z}} = X \otimes_{\mathbb{F}_1} \mathbb{Z}$ over $\mathbb{Z}$ and an immersion of affine gadgets $i\colon X \to \mathcal{G}(X_{\mathbb{Z}})$ [in particular, the points in the variety over $\mathbb{F}_1$ are points in $X_{\mathbb{Z}}$] satisfying the following universal property: for every affine variety V over $\mathbb{Z}$ and every morphism of affine gadgets $\varphi\colon X \to \mathcal{G}(V)$, there exists a unique algebraic morphism $\varphi_{\mathbb{Z}}\colon X_{\mathbb{Z}} \to V$ such that $\varphi = \mathcal{G}(\varphi_{\mathbb{Z}}) \circ i$, i.e. the diagram



commutes.

3.3. **Examples.**

Example 3.7. Any finite abelian group D defines an affine variety over $\mathbb{F}_1$, denoted $\operatorname{Spec}(D)$: the functor $\underline{\operatorname{Spec}(D)}$ is the functor represented by D, the algebra is $\mathbb{C}[D]$, and the evaluation is the obvious one.

Example 3.8. We define the multiplicative group $X = \mathbb{G}_m/\mathbb{F}_1$ as the triple $(\underline{X}, \mathcal{A}_X, e_X)$ where
(1) $\underline{X}(D) = D$,
(2) $\mathcal{A}_X$ is the algebra of continuous complex functions on the circle S^1, and
(3) if $P \in \underline{X}(D)$ and $f \in \mathcal{A}_X$, for every character $\chi\colon D \to \mathbb{C}^\times$, $f(P) \in \mathbb{C}[D]$ is such that $\chi(f(P)) = f(\chi(P))$.

Proposition 3.9. *$\mathbb{G}_m/\mathbb{F}_1$ is an affine variety over $\mathbb{F}_1$ such that* $\mathbb{G}_m \otimes_{\mathbb{F}_1} \mathbb{Z} = \operatorname{Spec}(\mathbb{Z}[T, T^{-1}])$.

Example 3.10. The affine line $\mathbb{A}^1/\mathbb{F}_1$ is defined as the triple $(\underline{X}, \mathcal{A}_X, e_X)$ by
(1) $\underline{X}(D) = D \amalg \{0\}$,
(2) $\mathcal{A}_X$ is the algebra of continuous functions on the closed unit disk which are holomorphic in the open unit disk, and

(3) if $P \in \underline{X}(D)$ and $f \in \mathcal{A}_X$, for any character $\chi\colon D \to \mathbb{C}^\times$, we have $\chi(f(P)) = f(\chi(P))$.

Proposition 3.11. *$\mathbb{A}^1/\mathbb{F}_1$ is an affine variety over $\mathbb{F}_1$ with extension of scalars* $\mathbb{A}^1 \otimes_{\mathbb{F}_1} \mathbb{Z} = \mathrm{Spec}(\mathbb{Z}[T])$.

4. Varieties over $\mathbb{F}_1$

4.1. **Definition.** To get varieties over $\mathbb{F}_1$ (and not only affine ones), we proceed again by analogy with Proposition 3.1. Let $\mathsf{Aff}_{\mathbb{F}_1}$ be the category of affine varieties over $\mathbb{F}_1$ (a full subcategory of the category of affine gadgets).

Definition 4.1. An *object over* $\mathbb{F}_1$ is a triple $X = (\underline{\underline{X}}, \mathcal{A}_X, e_X)$ consisting of

(1) a contravariant functor $\underline{\underline{X}}\colon \mathsf{Aff}_{\mathbb{F}_1} \to \mathsf{Set}$,
(2) a $\mathbb{C}$-algebra $\mathcal{A}_X$, and
(3) a natural transformation $e_X\colon \underline{\underline{X}} \Rightarrow \mathrm{Hom}(\mathcal{A}_X, \mathcal{A}_-)$.

Example 4.2. Assume V is an algebraic variety over $\mathbb{Z}$. Then we can define an object $X = \mathcal{O}b(V)$ as follows:

(1) $\underline{\underline{X}}(Y) = \mathrm{Hom}_{\mathbb{Z}}(Y_{\mathbb{Z}}, V)$,
(2) $\mathcal{A}_X = \Gamma(V_{\mathbb{C}}, \mathcal{O})$, and
(3) given $u \in \mathrm{Hom}_{\mathbb{Z}}(Y_{\mathbb{Z}}, V)$ and $f \in \mathcal{A}_X$, then $e_X(u)(f) = i^* u^*(f)$.

Morphisms and immersions of objects are defined as the corresponding notions for affine gadgets. Finally

Definition 4.3. A *variety over* $\mathbb{F}_1$ is an object $X = (\underline{\underline{X}}, \mathcal{A}_X, e_X)$ over $\mathbb{F}_1$ such that

(1) for any $D \in \mathsf{Ab}_f$, the set $\underline{\underline{X}}(\mathrm{Spec}(D))$ is finite;
(2) there exists a variety $X_{\mathbb{Z}} = X \otimes_{\mathbb{F}_1} \mathbb{Z}$ over $\mathbb{Z}$ and an immersion of objects $i\colon X \to \mathcal{O}b(X_{\mathbb{Z}})$ satisfying the following universal property: for every variety V over $\mathbb{Z}$ and every morphism of objects $\varphi\colon X \to \mathcal{O}b(V)$, there exists a unique algebraic morphism $\varphi_{\mathbb{Z}}\colon X_{\mathbb{Z}} \to V$ such that $\varphi = \mathcal{O}b(\varphi_{\mathbb{Z}}) \circ i$.

4.2. **Examples.** Any affine variety X over $\mathbb{F}_1$ is also a variety over $\mathbb{F}_1$: $\underline{\underline{X}}$ is the functor represented by X, $\mathcal{A}_X$ and e_X are the obvious ones.

The following proposition (see [8] Proposition 5) allows one to define a variety over $\mathbb{F}_1$ by glueing subvarieties.

Proposition 4.4. *Let V be a variety over $\mathbb{Z}$ and $V = \bigcup_{i\in I} U_i$ a finite open cover of V. Assume there is a finite family of varieties $X_i = (\underline{\underline{X}}_i, \mathcal{A}_i, e_i), i \in I$, and $X_{ij} = (\underline{\underline{X}}_{ij}, \mathcal{A}_{ij}, e_{ij}), i \neq j$, and immersions $X_{ij} \to X_i$ and $X_i \to \mathcal{O}b(V)$ of varieties over $\mathbb{F}_1$ such that*

(1) *$X_{ij} = X_{ji}$ and the composites $X_{ij} \to X_i \to \mathcal{O}b(V)$ and $X_{ij} \to X_j \to \mathcal{O}b(V)$ coincide;*
(2) *the maps $(X_{ij})_{\mathbb{Z}} \to (X_i)_{\mathbb{Z}}$ coincide with the inclusions $U_i \cap U_j \to U_i$, the maps $X_i \to \mathcal{O}b(V)$ induce the inclusions $U_i \to V$.*

For any affine variety Y over $\mathbb{F}_1$ define

$$\underline{\underline{X}}(Y) = \bigcup_i \underline{\underline{X}}_i(Y)$$

(union in $\mathrm{Hom}_{\mathbb{Z}}(Y_{\mathbb{Z}}, V)$*) and let*

$$\mathcal{A}_X = \left\{ (f_i) \in \prod_i \mathcal{A}_i \middle| f_i|_{X_{ij}} = f_j|_{X_{ij}} \right\}.$$

Then the object $X = (\underline{\underline{X}}, \mathcal{A}_X, e_X)$ (where e_X is the obvious evaluation) is a variety over $\mathbb{F}_1$ and $X \otimes_{\mathbb{F}_1} \mathbb{Z}$ is canonically isomorphic to V.

5. Zeta Functions

Let $X = (\underline{\underline{X}}, \mathcal{A}_X, e_X)$ be a variety over $\mathbb{F}_1$. We make the following assumption:

ASSUMPTION: There exists a polynomial $N(x) \in \mathbb{Z}[x]$ such that, for all $n \geq 1$, $\#\underline{\underline{X}}(\mathbb{F}_{1^n}) = N(n+1)$.

Consider the following series:

$$Z(q,T) = \exp\left(\sum_{r \geq 1} N(q^r) \frac{T^r}{r} \right).$$

Now take $T = q^{-s}$ to get a function of s and q. For every $s \in \mathbb{R}$, the function $Z(q, q^{-s})$ is meromorphic and has a pole at $q = 1$ of order $\chi = N(1)$. We let q go to 1 to get a zeta function over $\mathbb{F}_1$. We define

$$\zeta_X(s) = \lim_{q \to 1} Z(q, q^{-s})(q-1)^{\chi}.$$

Lemma 5.1. *If $N(x) = \sum_{k=0}^{d} a_k x^k$ then*

$$\zeta_X(s) = \prod_{k=1}^{d} (s-k)^{-a_k}.$$

Proof. We may assume that $N(x) = x^k$. Then we have

$$Z(q, q^{-s}) = \exp\left(\sum_{r \geq 1} q^{kr} \frac{q^{-rs}}{r} \right) = \exp(-\log(1 - q^{k-s})) = \frac{1}{1 - q^{k-s}}.$$

Now we have that

$$\lim_{q \to 1} \frac{q-1}{1 - q^{k-s}} = \frac{1}{s-k}.$$

□

For instance, if $X = \mathbb{G}_m/\mathbb{F}_1$, we get $\#\underline{\underline{X}}(\mathbb{F}_{1^n}) = n = N(n+1)$ with $N(x) = x - 1$. Therefore

$$\zeta_X(s) = \frac{s}{s-1}.$$

6. Toric Varieties over $\mathbb{F}_1$

6.1. **Toric varieties.** Let $d \geq 1$, $N = \mathbb{Z}^d$, and $M = \mathrm{Hom}(N, \mathbb{Z})$. Let $N_\mathbb{R} = N \otimes_\mathbb{Z} \mathbb{R}$ and $M_\mathbb{R} = M \otimes_\mathbb{Z} \mathbb{R}$. We then have the duality pairing

$$\langle \cdot, \cdot \rangle \colon M_\mathbb{R} \times N_\mathbb{R} \to \mathbb{R}\,.$$

Definition 6.1. A *cone* is a subset $\sigma \subset N_\mathbb{R}$ of the form

$$\sigma = \sum_{i \in I} \mathbb{R}_+ n_i$$

where $(n_i)_{i \in I}$ is a finite family in N.

We define the *dual* and the *orthogonal* of σ by

$$\sigma^* = \left\{ v \in M_\mathbb{R} \middle| \langle v, x \rangle \geq 0 \text{ for all } x \in \sigma \right\} \text{ and}$$
$$\sigma^\perp = \left\{ v \in M_\mathbb{R} \middle| \langle v, x \rangle = 0 \text{ for all } x \in \sigma \right\}$$

respectively.

A cone is *strict* if it does not contain any line.

A *face* is a subset $\tau \subset \sigma$ such that there is a $v \in \sigma^*$ with $\tau = \sigma \cap v^\perp$.

Definition 6.2. A *fan* is a finite collection $\Delta = \{\sigma\}$ of strict cones such that
(1) if $\sigma \in \Delta$, any face of σ is in Δ, and
(2) if $\sigma, \sigma' \in \Delta$, then $\sigma \cap \sigma'$ is a face of σ and σ'.

Definition 6.3. Given Δ, we define a variety $\mathbb{P}(\Delta)$ over $\mathbb{Z}$ as follows: for all $\sigma \in \Delta$, consider the monoid $S_\sigma = M \cap \sigma^*$. Set

$$U_\sigma = \mathrm{Spec}(\mathbb{Z}[S_\sigma]).$$

If $\sigma \subset \tau$, we have $U_\sigma \subset U_\tau$. The variety $\mathbb{P}(\Delta)$ is obtained by glueing the affine varieties U_σ, $\sigma \in \Delta$, along the subvarieties $U_{\sigma\cap\tau}$.

We assume that Δ is *regular*, i.e. any $\sigma \in \Delta$ is spanned by a subset of a basis of N. We shall define a variety $X(\Delta)$ over $\mathbb{F}_1$ such that $X(\Delta) \otimes_{\mathbb{F}_1} \mathbb{Z} = \mathbb{P}(\Delta)$.

6.2. **The affine case.** First, let us fix $\sigma \in \Delta$. For any $m \in S_\sigma$, let $\chi^m \colon U_\sigma \to \mathbb{A}^1$ be the function defined by m. When D is a finite abelian group, we define $\underline{X}_\sigma(D) \subset U_\sigma(\mathbb{Z}[D])$ to be the set of points P such that for any $m \in S_\sigma$, $\chi^m(P) \in D \amalg \{0\}$.

Let

$$C_\sigma = \left\{ x \in U_\sigma(\mathbb{C}) \middle| |\chi^m(x)| \leq 1 \text{ for all } m \in S_\sigma \right\} \text{ and}$$
$$\mathring{C}_\sigma = \left\{ x \in C_\sigma \middle| |\chi^m(x)| < 1 \text{ for all } m \in S_\sigma \text{ with } \langle m, \sigma \rangle \neq 0 \right\}.$$

We define $\mathcal{A}_\sigma$ to be the ring of continuous functions $f \colon C_\sigma \to \mathbb{C}$ such that $f|_{\mathring{C}_\sigma}$ is holomorphic . Finally, if $P \in \underline{X}_\sigma(D)$, $f \in \mathcal{A}_\sigma$ and $\chi \colon D \to \mathbb{C}^\times$, we define $e_\sigma(P)$ by the formula $\chi(e_\sigma(P)(f)) = f(\chi(P))$.

The following is a generalization of Proposition 3.9. and Proposition 3.11.

Proposition 6.4. *If σ is regular, then $X_\sigma = (\underline{X}_\sigma, \mathcal{A}_\sigma, e_\sigma)$ is an affine variety over $\mathbb{F}_1$ such that $X_\sigma \otimes_{\mathbb{F}_1} \mathbb{Z} = U_\sigma$.*

Proof. Suppose $\{n_1, \dots, n_d\}$ is a basis for N and that $\sigma = \mathbb{R}_+ n_1 + \cdots \mathbb{R}_+ n_{d-r}$. Let $\{m_1, \dots, m_d\}$ be the dual basis of M. Then

$$S_\sigma = \mathbb{N}m_1 + \cdots \mathbb{N}m_{d-r} + \mathbb{Z}m_{d-r+1} + \cdots \mathbb{Z}m_d = M \cap \sigma^*\,,$$

and as $U_\sigma(\mathbb{C}) = \mathbb{C}^{d-r} \times (\mathbb{C}^\times)^r$, we have

$$C_\sigma = \left\{ x \in U_\sigma(\mathbb{C}) \big| |x_1|, \dots, |x_{d-r}| \leq 1 \text{ and } |x_{d-r+1}| = \cdots = |x_r| = 1 \right\} \text{ and}$$
$$\mathring{C}_\sigma = \left\{ x \in U_\sigma(\mathbb{C}) \big| |x_1|, \dots, |x_{d-r}| < 1 \text{ and } |x_{d-r+1}| = \cdots = |x_r| = 1 \right\}.$$

Furthermore

$$\underline{X}_\sigma(D) = (D \amalg \{0\})^{d-r} \times D^r.$$

Let V be an affine variety over $\mathbb{Z}$, and let $\varphi\colon X_\sigma \to \mathcal{G}(V)$ be a morphism of affine gadgets. We must find a $\varphi_\mathbb{Z}\colon U_\sigma \to V$ such that $\varphi = \mathcal{G}(\varphi_\mathbb{Z}) \circ i$. This is the same as a morphism from the algebra of functions on V to the algebra of functions on U_σ. Let $f \in \Gamma(V, \mathcal{O}_V)$. Then f induces a function $f_\mathbb{C}$ on the complex variety $V_\mathbb{C}$, and we may pull back this function to get a function on X_σ: $g_\mathbb{C} = \varphi^*(f_\mathbb{C}) \in \mathcal{A}_\sigma$. We must show that $g_\mathbb{C}$ is algebraic over $\mathbb{Z}$, i.e. that it comes from a $g \in \mathcal{O}(U_\sigma)$. Restrict $g_\mathbb{C}$ to $(S^1)^d$, and look at the Fourier expansion

$$g_\mathbb{C}(\exp(2\pi i\theta_1), \dots, \exp(2\pi i\theta_d)) = \sum_{J \in \mathbb{Z}^d} c_J \exp(2\pi i(J\cdot\theta)) \text{ where } J\cdot\theta = \sum_{k=1}^d j_k\theta_k.$$

Since $g_\mathbb{C}$ is holomorphic on C_σ, we must have $c_J = 0$ when $j_k < 0$ for $1 \leq k \leq d-r$. We want to show that $g_\mathbb{C}$ is an integral polynomial in the first $d-r$ coordinates and an integral Laurent polynomial in the r remaining coordinates, i.e.

$$g_\mathbb{C} \in \mathbb{Z}[T_1, \dots, T_{d-r}, T_{d-r+1}^{\pm 1}, \dots, T_d^{\pm 1}].$$

Let $n > 1$, and consider $D = (\mathbb{Z}/n)^d$. Then if

$$P_k = \underbrace{(0, \dots, 0, 1, 0, \dots, 0)}_{k^{\text{th}} \text{ slot is } 1},$$

we get a point $P = (P_1, \dots, P_d) \in D^d \subset \underline{X}_\sigma(D)$. For $a = (a_k) \in D$, define $\chi_a\colon D \to \mathbb{C}^\times$ by

$$\chi_a(b) = \prod_{k=1}^d \exp\left(2\pi i \frac{a_k b_k}{n}\right).$$

Then, as φ commutes with evaluations, we get

$$\begin{aligned} \chi_a(e_\sigma(P)(g_\mathbb{C})) &= g_\mathbb{C}(\chi_a(P)) = g_\mathbb{C}(\exp(2\pi i a_1/n), \dots, (\exp(2\pi i a_d/n)) \\ &= \chi_a(f(\underline{\varphi}(P))) = \chi_a(Q) \end{aligned}$$

where $Q = f(\underline{\varphi}(P)) \in f(V(\mathbb{Z}[D])) \subset \mathbb{Z}[D]$. The Fourier coefficients of $g_{\mathbb{C}}$ are given by the formula

$$\begin{aligned} c_J &= \int_{(S^1)^d} g_{\mathbb{C}}(\exp(2\pi i\theta_1), \ldots, \exp(2\pi i\theta_k)) \exp(-2\pi i(J\cdot\theta)) d\theta_1 \cdots d\theta_d \\ &= \lim_{n\to\infty} n^{-d} \sum_a g_{\mathbb{C}}(\exp(2\pi i a_1), \ldots, \exp(2\pi i a_k)) \exp(-2\pi i(J\cdot a)/n) \\ &= \lim_{n\to\infty} n^{-d} \sum_a \chi_a(Q) \exp(-2\pi i(J\cdot a)/n). \end{aligned}$$

But as $Q \in \mathbb{Z}[D]$ we must have, for every n,

$$n^{-d} \sum_a \chi_a(Q) \exp(-2\pi i(J\cdot a)/n) \in \mathbb{Z}\,.$$

Therefore $c_J \in \mathbb{Z}$, and $c_J = 0$ for almost all J, as desired. □

6.3. **The general case.** Let Δ be a regular fan. For every affine variety Y over $\mathbb{F}_1$ let

$$\underline{\underline{X}}_{\Delta}(Y) = \bigcup_{\sigma\in\Delta} \mathrm{Hom}(Y, X_\sigma)\,.$$

Define

$$C_\Delta = \bigcup_{\sigma\in\Delta} C_\sigma \subset \mathbb{P}(\Delta)(\mathbb{C})\,,$$

and let $\mathcal{A}_\Delta$ be the algebra of continuous functions $f\colon C_\Delta \to \mathbb{C}$ such that, for all $\sigma \in \Delta$, the restriction of f to $\mathring{C}_\sigma$ is holomorphic. Finally, if $P \in \mathrm{Hom}(Y, X_\sigma) \subset \underline{\underline{X}}_{\Delta}(Y)$ and $f \in \mathcal{A}_\Delta$, define $e_\Delta(P)(f) = P^*(f) \in \mathcal{A}_Y$.

The following is a consequence of Proposition 4.4 and Proposition 6.4.

Theorem 6.5. *The object $X(\Delta) = (\underline{\underline{X}}_{\Delta}, \mathcal{A}_\Delta, e_\Delta)$ over $\mathbb{F}_1$ is a variety over $\mathbb{F}_1$ such that*

$$X(\Delta) \otimes_{\mathbb{F}_1} \mathbb{Z} = \mathbb{P}(\Delta)\,.$$

Remark 6.6. There exists $N(x) \in \mathbb{Z}[x]$ such that, for all $n \geq 1$, $\#X_\Delta(\mathbb{F}_{1^n}) = N(n+1)$.

7. Euclidean Lattices

Let Λ be a free $\mathbb{Z}$-module of finite rank, and $\|\cdot\|$ an Hermitian norm on $\Lambda\otimes_{\mathbb{Z}}\mathbb{C}$. We view $\overline{\Lambda} = (\Lambda, \|\cdot\|)$ as a vector bundle on the complete curve $\mathrm{Spec}(\mathbb{Z}) \amalg \{\infty\}$. The finite pointed set

$$H^0(\mathrm{Spec}(\mathbb{Z}) \amalg \{\infty\}, \overline{\Lambda}) = \left\{ s \in \Lambda \middle| v_\infty(s) = -\log\|s\| \geq 0 \right\} = \Lambda \cap B\,,$$

where $B = \left\{ v \in \Lambda\otimes_{\mathbb{Z}}\mathbb{C} \middle| \|v\| \leq 1 \right\}$, is viewed as a finite dimensional vector space over $\mathbb{F}_1$.

We can define an affine variety over $\mathbb{F}_1$ as follows. We let

$$\underline{X}(D) = \left\{ P = \sum_{v\in\Lambda\cap B} v\otimes\alpha_v \middle| \alpha_v \in D \right\} \subset \Lambda\otimes_{\mathbb{Z}} \mathbb{Z}[D]\,.$$

If $\Lambda_0 \subset \Lambda$ is the lattice spanned by $\Lambda \cap B$ we consider

$$C = \left\{ v \in \Lambda_0 \otimes_{\mathbb{Z}} \mathbb{C} \middle| \|v\| \leq \mathrm{card}(V \cap B) \right\} ,$$

and we define $\mathcal{A}_X$ as the algebra of continuous functions $f : C \to \mathbb{C}$ such that $f|_{\mathring{C}}$ is holomorphic. Finally, for each $D \in \mathsf{Ab}_f$, $P \in \underline{X}(D)$, $f \in \mathcal{A}_X$, and $\chi : D \to \mathbb{C}^\times$, we define

$$\chi(f(P)) = f\left(\sum_{v \in \Lambda \cap B} \chi(a_v) v \right) .$$

Proposition 7.1.

(1) *The affine gadget* $X = (\underline{X}, \mathcal{A}_X, e_X)$ *is an affine variety over* $\mathbb{F}_1$ *such that* $X \otimes_{\mathbb{F}_1} \mathbb{Z} = \mathrm{Spec}(\mathrm{Symm}_{\mathbb{Z}}(\Lambda_0^*))$.

(2) *There is a polynomial* $N \in \mathbb{Z}[x]$ *such that, for all* $n \geq 1$, $\#X(\mathbb{F}_{1^n}) = N(2n+1)$.

This proposition raises the question whether there is a way to attach to $\overline{\Lambda}$ a torified variety in the sense of [6].

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