# Leibniz in Cantor's Paradise

## A Dialogue on the Actual Infinite

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May 23, 2017

*Dialogus de anima brutorum inter Pythagoram et Cartesium in Elysiis campis sibi obviam factos*. (Leibniz, 12 December 1676: A VI, iii 582)

## Introduction:

In circumstances too bizarre to relate here, I recently chanced upon the following conversation between Georg Cantor and Gottfried Leibniz, apparently on the occasion of their first encounter in the Afterlife. It seemed safe to assume that it would be worth recording for posterity and, looking at the transcript now, that assumption seems to be borne out. Rather surprisingly, Leibniz was by no means the naive understudy one might have supposed he would be: he sets out a consistent conception of the actual infinite and poses some pointed objections to Cantor's interpretation of it as transfinite.

Although I was unable to begin my recording immediately, missing the many 'sehr geehrter Herr Professor's and other extravagant greetings and praises that went back and forth, my transcription (and translation from the mix of Latin, German and French in which they spoke) begins immediately afterwards.

*Cantor*: I always remember with joy what you said of the *actual infinite*, that you were so much in favour of it that instead of admitting that Nature abhors it, as is commonly said, you hold that Nature affects it everywhere, the better to indicate the perfections of its Author.[1] This remark gave me great inspiration!

*Leibniz*: I am very glad. Indeed, I believe that there is no part of matter which is not, I do not say divisible, but actually divided; and that consequently the least particle ought to be considered as a world full of an infinity of different creatures.[2]

[1] Leibniz, Letter to Foucher (January 1692), *Journal des Sçavans*, March 16, 1693, GP I 416. This is my translation, as are all others from Leibniz's Latin and French, except where noted. All translations from his early unpublished work are from R. T. W. Arthur, trans. and ed., *G. W. Leibniz: The Labyrinth of the Continuum* (New Haven: Yale University Press, 2001), hereafter abbreviated *LoC*. Throughout this piece I have set the actual words of the participants in a different font and colour. Some inessentials, such as tense, have been altered to ease the fit. I have mixed up quotations from different periods of Leibniz's work, as I believe his views on the infinite underwent no substantive change during the last forty of so years of his life; the consistency of the resulting pastiche must serve as my only warrant for this here.
[2] *Ibidem.* Cantor quotes this whole passage from p. 118 of Erdmann's collection of Leibniz's works in his "Grundlagen einer allgemeinenen Mannigfaltigkeitslehre" (Leipzig, 1883), in Georg Cantor, *Gesammelte Abhandlungen* (hereafter *GA*), ed. Ernst Zermelo (Berlin, 1932; repr. Hildesheim: Georg Olms, 1962) p. 179. For my understanding of Cantor's philosophy I am indebted to J. Dauben, *Georg Cantor: his mathematics and philosophy of the infinite* (Boston: Harvard University Press, 1979), (hereafter 'Dauben'), and even more deeply to Michael Hallett,

*Cantor*: Here again I took inspiration from your ideas,[3] for I believe that you were correct in your opposition to the Newtonian philosophy, and in pursuing a far-reaching *organic* explanation of nature.[4] I fully agree with you that there is an actually infinite number of created individual essences, not only in the universe but also here on our Earth, and, in all likelihood, even in every extended part of space however small.[5] I too hold that in order to obtain a satisfactory *explanation of nature*, one must posit the ultimate or properly *simple* elements of matter to be *actually infinite in number*.[6] In agreement with you, I call these simple elements of nature *monads* or *unities*. But since there are *two specific, different types of matter interacting with one another*, namely *corporeal matter* and *aetherial matter*, one must also posit two different classes of *monads* as foundations, *corporeal monads* and *aetherial monads*. From this standpoint the question is raised (a question that occurred neither to you nor to later thinkers): what *power* is appropriate to these types of matter with respect to their elements, insofar as they are considered *sets* of *corporeal* and *aetherial monads*. In this connection, I posit the corporeal monads, as discrete unities, to be as many as the natural numbers; and the aether, as continuous, to be composed of aetherial monads equinumerous with the points on a line. That is, I frame the hypothesis that the *power* of the corporeal monads is (what I call in my researches) the *first* power, i.e. $\aleph_0$, the first of the transfinite cardinal numbers; whilst the power of aetherial matter is the *second*, i.e. I posit the number of aetherial monads to be equal to $\aleph_1$, the second cardinal number, which I believe (but have never been able to prove) to be the power of the continuum.[7]

*Leibniz*: I am most flattered that you honour me as the inspiration for your views. I also deeply sympathize with your opposition to the materialism implicit in the Newtonian philosophy, and your attempts to show the superiority of the organic philosophy I

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*Cantorian set theory and the limitation of size* (Oxford: Clarendon Press, 1984), (hereafter 'Hallett'). All translations from Cantor's German are my own, except where from Hallett or Dauben as noted.

3 Cf. Dauben, p. 292: "Leibniz was the source of greatest inspiration, as Cantor made clear by his choice of terminology: he called the ultimate components of matter which he hypothesized 'monads'" (292).

4 Cantor, Letter to Valson, Dauben n. 85, p. 295; cf. Cantor, Über, *GA* 177.

5 Cantor, Letter to Cardinal Franzelin, 22 January 1886; *GA* 399.

6 This and all remaining quotations in this paragraph are culled from Cantor, "Über verschiedene Theoreme aus der Theorie der Punktmengen in einem *n*-fach ausgedehnten steigen Raume $G_n$" (1885), *GA* 275-276.

7 *Ibid.*, *GA* 276. Cf. Dauben, p. 292.

have always favoured. So it is with all the greater regret that I have to say that I must withhold my assent from these hypotheses of yours. For, setting aside my dissent from the notions that a continuum can be an actual existent, and that monads can compose a material continuum, it seems our deepest disagreement lies in the nature of the actual infinite itself. For where you believe in the existence of actually infinite numbers of monads I, on the contrary, believe that there is no such thing as infinite number, nor an infinite line nor any other infinite quantity, if one takes them as true wholes.[8]

*Cantor*: I am aware of your refusal to countenance infinite number. But I must confess I have never understood your position. For even though I found many places in your works where you come out against infinite numbers, I was still in the happy position of being able to find other pronouncements of yours where, seemingly in contradiction to this, you declare yourself to be unequivocally *for* the actual infinite (as distinct from the Absolute),[9] as in the case of the infinitude of substances we have already noted. But surely if there are actually infinitely many monads then there must be an infinite number of them! And what about your Infinitesimal Calculus? You must agree that this is a testament to the necessity of infinite number for mathematics!

*Leibniz*: On the contrary, in spite of my *Infinitesimal Calculus*, I admit no genuine (*véritable*) infinite number, even though I confess that the multiplicity (*multitude*) of things surpasses every finite number, or rather every number.[10] When it is said that there is an infinity of terms, it is not being said that there is some specific number of them, but that there are more than any specific number.[11] So, I understand a quantity to be *infinite* provided it is greater than any that can be assigned by us or designated by numbers.[12] Thus, properly speaking, it is true that there is an infinity of things, if this is understood in the sense that there are always more of them than can be assigned.[13]

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[8] Quoted from Cantor, "Grundlagen" (1883), *GA* 179.
[9] *Ibidem.*
[10] Leibniz, Letter to Samuel Masson, 1716, GP VI 629.
[11] Leibniz, Letter to Johann Bernoulli, 1699, GM III 566.
[12] Leibniz, *De Quadratura Arithmetica*, 1676 (in the Critical Edition of Eberhard Knobloch, Göttingen: Vandenhoek & Ruprecht, 1993), p. 133.
[13] Leibniz, *New Essays on Human Understanding* (1704; trans. & ed. Peter Remnant and Jonathan Bennett, Cambridge University Press, 1981), chapter xvii: 'Of infinity'; A VI 6,157.

*Cantor*: Here, it seems to me, you are following Aristotle. For he claimed that this was how mathematicians understood the infinite, as just a kind of potential for indefinite extensibility: "In point of fact, they neither need the infinite nor use it, but need only posit that a finite line may be produced as far as they wish" (*Physics*, 207b 31-32). Likewise he allowed that one can have an infinite by division, since this amounts to saying only that no matter how many divisions one makes it is always possible to make more. But according to him this means that "this infinite is potential, never actual" (*Physics*, 207b 12-13). However, you wish to claim that there is an actual infinity of divisions, not merely a potential one, or an indefinite one, as Descartes would say.[14]

*Leibniz*: Regarding your last remark, I have always held that Descartes' "indefinite" is not in the thing, but the thinker.[15] In any case, contrary to his own recommend-ation in Part I of his *Principles* that instead of the term 'infinite' we use the term 'indefinite', or that whose limits cannot be found by us, in Part II of the same work ( §36 ) Descartes himself admits matter to be really divided by motion into parts that are smaller than any assignable, and therefore actually infinite.[16] Aristotle's views are more subtle. I think he is right to hold both that there is no infinite number, nor any infinite line or other infinite quantity, yet that there is an infinite by division, provided this is taken in the sense that there are so many divisions that there are always more of them than can be assigned. The Scholastics were taking that view, or should have been, when they allowed a *syncategorematic infinite*, as they called it, but not a *categorematic* one.[17] That is, to say that matter is actually infinitely divided is to say that there are infinitely many actual divisions, and this

[14] Cantor equates the potential infinite with the indefinite at *GA* 373. See note 20 below.
[15] Leibniz, *Theory of Abstract Motion* (1671), §1, (*LoC* 339). O. Bradley Bassler, in his admirable article "Leibniz on the Indefinite as Infinite" (*The Review of Metaphysics*, 51, (June 1998): 849-874, n. 35), argues that this is an early view that Leibniz comes to reject. I demur: the parts of the continuum, it is true, Leibniz does come to regard as indeterminate. But the parts of matter are always for him determinate, and not at all indefinite; they are the result of an actually infinite division. This is a view I believe he never relinquishes from 1670 till his death.
[16] Leibniz, "Notes on Descartes' *Principles*", *LoC*, p. 25 (A VI 3, 214).
[17] Leibniz, *New Essays*, A VI 6, 157. Likewise, in his letter to Des Bosses of September 1, 1706 (GP II 314-15) Leibniz says: "*There is a syncategorematic infinite* ... But *there is no categorematic infinite*, or one actually having infinite parts formally". But one could certainly be forgiven for identifying this syncategorematic infinite with the potential infinite on the basis of this letter, for Leibniz defines it here as a "passive potential for having parts, namely the possibility of dividing, multiplying, subtracting, adding". See O. B. Bassler's erudite footnote on the syncategorematic and categorematic in his "Leibniz on the Indefinite as Infinite", p. 855, n. 15, and the references cited therein.

"infinitely many" is understood in the syncategorematic sense that "there are not so many divisions that there are not more" (*non sunt tot quin sint plura*).[18] Indeed, accurately speaking, instead of an infinite number, we ought to say that there are more than any number can express, and instead of an infinite straight line that it is a straight line continued beyond any magnitude that can be assigned, so that a larger and larger straight line is always available.[19]

*Cantor*: With all due respect to the distinctions of the medieval schoolmen, this "syncategorematic infinite" appears to be nothing different from what I call the potential infinite.[20] The potential infinite means nothing other than an undetermined, variable quantity, always remaining finite, which has to assume values that either become smaller than any finite limit no matter how small, or greater than any finite limit no matter how great.[21] As an example of the latter, where one has an undetermined, variable finite quantity increasing beyond all limits, we can think of the so-called time counted from a determinate beginning moment, whereas an example of a variable finite quantity which decreases beneath every finite limit of smallness would be, for example, the correct presentation of your so-named differential.[22] There is no doubt that we cannot do without variable quantities in the sense of the potential infinite. But from this very fact the necessity of the actual infinite can be demonstrated.[23]

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[18] For this quotation and an illuminating discussion see Philip Beeley, *Kontinuität und Mechanismus* (Stuttgart: Franz Steiner, 1996), esp. pp. 59-60., etc. According to Beeley, Gregory of Rimini also held that "every continuum has a plurality of parts, and not so many finite in number that there are not more (*non tot finitas numero quin plures*), and has all its parts actually and at the same time" (p. 59); this is the syncategorematic infinite (although Gregory also upheld a categorematic infinity). Peter Geach, in his comments on an essay of Abraham Robinson's (*Problems in the Philosophy of Mathematics*, ed. Imre Lakatos, North Holland: Amsterdam, 1967, p. p. 41-42), points out that the high Scholastic meaning of 'syncategorematic' pertained to a certain way of using words, and sharply distinguishes the infinite in the syncategorematic sense as "there are infinitely many" from its assimilation to the potential infinite by late scholastics like Suàrez (and, following them, Leibniz: see his letter to Des Bosses quoted in the previous footnote). But Leibniz's position is, I argue here, consistent with the more precise use of the term, as by Ockham.

[19] Leibniz to Des Bosses, 11 March 1706, GP II 304-05.

[20] Cantor equates the potential infinite with the syncategorematic in "Über die verschiedenen Standpunkte in bezug auf das aktuelle Unendliche", (November 1885, *GA* 370-77): "That the so-called *potential* or *syncategorematic infinite* (Indefinitum) gives basis for no such division" [gibt zu keiner derartigen Einteilung Veranlassung..."] (*GA* 373).

[21] Cantor, "Mitteilungen" (1887-8), *GA* 409.

[22] Cantor, *ibid.*, *GA* 401; freely adapted. Compare Hallett's translation, p. 12.

[23] Cantor, "Über die verschiedenen Ansichten in Bezug auf die actualunendlichen Zahlen" (*Bihand Till Koniglen Svenska Vetenskaps Akademiens Handigar* **11** (19), 1-10 (1886), p. 9; adapted from Hallett's translation, p. 25.

*Leibniz*: I would like to see the demonstration.

*Cantor*: Well, in order for there to be a variable quantity in some mathematical study, the domain of its variability must strictly speaking be known beforehand through a definition. However, this domain cannot itself be something variable, since otherwise each fixed support for the study would collapse. Thus this domain is a definite, actually infinite set of values. Hence each potential infinite, if it is rigorously applicable mathematically, presupposes an actual infinite.[24]

*Leibniz*: This claim that the potential infinite presupposes an actual infinite was, as I am sure you know, already made by that eminent French mathematician, Blaise Pascal. He claimed that the world could not be potentially infinite in spatial extent without this presupposing an actual infinite. But since our finite minds cannot entertain such an actual infinite without falling into contradiction, and yet we have this demonstration of its existence, he took this to show that there are truths whose comprehension lies beyond the reach of finite minds. As I have argued, however, there is a way of understanding the actual infinite that is free of such contradiction, although I do not accept that this requires us to embrace actually infinite number, or a determinate infinite collection of values. This syncategorematic infinite, however should not be confused with the true or hypercategorematic infinite. The true [*veritable*] *infinite* is not found in a whole composed of parts. However, it is found elsewhere, namely in the *Absolute*, which is without parts, and which has influence on compound things since they result from the limitation of the absolute.[25]

*Cantor*: You will not find an opponent in me concerning the Absolute, which I agree cannot in any way be added to or diminished. It is therefore to be looked upon quantitatively as an absolute maximum. In a certain sense it transcends the human power of comprehension, and in particular is beyond mathematical determination.[26] But in this it is distinguished from that infinite which I call the

[24] *Ibidem.*
[25] "Quelques remarques sur le livre de Mons. Lock intitulé *Essay of Understanding*," 1695?-7, A VI 6, 7. In the *New Essays*, Leibniz writes: "The true infinite, strictly speaking, is only in the *absolute*, which precedes all composition and is not formed by the addition of parts" (A VI 6, 157); and to Johann Bernoulii he writes in 1698: "The real infinite is perhaps the absolute itself, which is not made up of parts, but which includes beings having parts eminently and in proportion to their degree of perfection." (GP III 499-500).
[26] Cantor, "Mitteilungen" (1887-8), *GA* 405; quoted in Hallett's translation, p. 13.

Transfinite. This is in itself constant, and larger than any finite, but nevertheless unrestricted, increasable, and in this respect thus bounded. Such an infinite is in its way just as capable of being grasped by our restricted understanding as is the finite in its way.[27]

*Leibniz*: I too have always distinguished the Immensum from the Unbounded, and that to which nothing can be added from that which exceeds any assignable number.[28]

*Cantor*: This corresponds to my own view, if by *assignable* you mean *finite*. For by the actual infinite I understand a quantum that surpasses in magnitude every finite quantity of the same kind.[29]

*Leibniz*: This puts me in mind of a distinction I used to make among three different degrees of infinity. In descending order, they are 1) the *absolutely infinite*, 2) the *maximum*, or greatest of its kind, and 3) the *mere infinite*. The first or absolutely infinite is that which contains everything.[30] This kind of infinite is in God, since he is all one and contains the requisites for existing of all else.[31] I believe we in are agreement on this kind of infinite.

*Cantor*: We are indeed. What surpasses all that is finite and transfinite is the single completely individual unity in which everything is included, which by many is called God.[32] But I am not sure we agree about the second, since I believe the only maximum is God himself. But do finish your explanation.

*Leibniz*: Well, by the *maximum* I mean that to which nothing can be added: for instance, a line unbounded on both sides, which is obviously infinite, since it contains every length;[33] or the whole of space, which is the greatest of all extended things.[34] But perhaps our views do not differ as much as you think, for I agree with you that there is no maximum in things. For I hold that two notions that are [equally]

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[27] Cantor, *Nachlass* VI, p. 99; quoted from Hallett, p. 14.
[28] Leibniz, "Annotated Excerpts from Spinoza", *LoC*, p. 115 (A VI.iii.281).
[29] Cantor, "Mitteilungen" (1887-8), *GA* 401. This passage is quoted more fully below.
[30] Leibniz, "Annotated Excerpts from Spinoza", *LoC*, p. 115 (A VI.iii.282).
[31] Leibniz, "On Spinoza and On the Infinite", *LoC*, p. 43 (A VI.iii.385).
[32] Cantor, Letter to G. C. Young, June 20, 1908; quoted from Dauben, p. 290.
[33] Leibniz, "Annotated Excerpts from Spinoza", *LoC*, p. 115 (A VI.iii.282).
[34] Leibniz, "On Spinoza and On the Infinite", *LoC*, p. 43 (A VI.iii.385).

excluded from the realm of intelligibles, are that of a minimum, and that of a maximum: what lacks parts, and what cannot be part of another.[35]

Lastly, there is the *infinite in lowest degree*,[36] which I usually call the *mere infinite*,[37] which is that whose magnitude is greater than we can expound by an assignable ratio to sensible things. An example would be the infinite space comprised between Apollonius's Hyperbola and its asymptote, which is one of the most moderate of infinities, to which there somehow corresponds in numbers the sum of this space: $^1/_1 + ^1/_2 + ^1/_3 + ^1/_4 + \ldots$, which is $^1/_0$. Only instead of this 0, or nought, let us rather understand instead a quantity infinitely or unassignably small, which is greater or smaller according as we have assumed the last denominator of this infinite series of fractions (which is itself also infinite) smaller or greater. For a maximum does not apply in the case of numbers.[38]

*Cantor*: I take exception to the idea that the common infinite is greater than expressible by a ratio to anything sensible: one must distinguish between numbers as they are in and for themselves, and in and for the Divine Intelligence, and how these same numbers appear in our restricted, discursive comprehension.[39] Also, I am glad we agree that a maximum does not apply to numbers, and I even agree with you that there is no number of all numbers. But regarding natural numbers, I do not believe that their infinity is "unassignable". I submit that one can indeed assign a number to all the natural numbers, just as one can assign a limit to a converging infinite sequence even when there is no last number in that sequence. In each case it is the *first* whole number which follows all the numbers.[40]

*Leibniz*: I remain to be convinced that one can assign such numbers. I concede the infinite multiplicity of terms, but this multiplicity does not constitute a number or a single whole. It signifies nothing but that there are more terms than can be designated by a number. Just so, there is a multiplicity or complex of numbers, but this multiplicity is not a number or a single whole.[41]

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35 Leibniz, "On Minimum and Maximum", *LoC*, p. 13 (A VI.iii.98).
36 Leibniz, "Annotated Excerpts from Spinoza", *LoC*, p. 115 (A VI.iii.282).
37 Leibniz, "On Spinoza and On the Infinite", *LoC*, p. 43 (A VI.iii.385).
38 Leibniz, "Annotated Excerpts from Spinoza", *LoC*, p. 115 (A VI.iii.282).
39 Hallet? reference lost! try: Cantor, *Nachlass* VI, p. 99; quoted from Hallett, p. 14.
40 Cantor, "Grundlagen", *GA* 195.
41 Leibniz, Letter to Bernoulli, GM.iii.575. On Leibniz's philosophy of mathematics, the reader is referred to the excellent discussion by Samuel Levey, "Leibniz's Constructivism and Infinitely

*Cantor*: You are hardly alone, although lately the weight of opinion seems to have come around to my way of thinking. But in my view all so-called proofs against the possibility of infinite number are faulty. As can be shown in every particular case, and also concluded on general grounds, their chief failing, and where their πρωτον ψευδος [initial mistake] lies, is that from the outset they expect or even impose all the properties of finite numbers upon the numbers in question.[42] As an example, consider the following argument by Tongiorgi:

> Let us suppose that an infinite multiplicity is constructed by the accumulation of ones. This will be an infinite number, and will be equal to *the number which immediately precedes it* with one added to it. Now was *the preceding number* infinite, or was it not? You cannot say it is infinite, for it could be further increased, and was in fact increased by the addition of one. Therefore it was finite, and with one added, it became infinite. Yet from two finites an infinite has emerged; which is absurd. (§350; $2^0$ Pesch §412, $3^0$, $4^0$)

Here it is falsely assumed (because one is accustomed to this with finite numbers) that an actually infinite number must necessarily have a whole number that is the first to precede it, from which it results by the addition of one.[43] But according to my theory of the transfinite, the smallest transfinite ordinal number ω is preceded by all the finite ordinals, which have no maximum. Thus this actually infinite number is not a natural number (or "inductive number", as Russell called it), since it succeeds all natural numbers and is regarded as the *limit* of those numbers, *i.e.* is defined as the next greater number than all of them.[44]

*Leibniz*: I can appreciate why you would want to exclude actually infinite number from the sequence of all natural numbers. For I once thought I could prove that the number of finite numbers cannot be infinite[45] by a similar proof as follows: If numbers can be assumed as continually exceeding each other by one, the number of such finite numbers cannot be infinite, since in that case the number of numbers

Folded Matter" pp. 134-162, *New Essays on the Rationalists*, ed. Rocco Gennaro and Charles Huenemann (New York, Oxford University Press, 1999). He quotes this passage on p. 139.

[42] Cantor, "Über die verschnieden Standpunkte…", *GA* 371-2; cf. Dauben, 1979, 125; Cantor, "Mitteilungen", *GA* 396.

[43] Cantor, "Mitteilungen" (1887-8), *GA* 394.

[44] Cantor, "Grundlagen (1883)", §11, *GA* 196.

[45] Leibniz, "On the Secrets of the Sublime", *LoC*, p. 51 (A VI.iii.477).

is equal to the greatest number, which is supposed to be finite.[46] But then I realized that to this argument it must be responded that there is no such thing as the greatest number;[47] this kind of demonstration proves only that such a series is unbounded.[48]

*Cantor*: But this is music to my ears! So many of the so-called finitists have failed to appreciate this unboundedness, and have assumed that if the natural numbers are actually infinite, there must be a greatest such number.[49]

*Leibniz*: Yet it seems to me that a similar argument could be applied to the idea that there could be a greatest number even if the parts are *actually infinite in number*. For suppose I accept that numbers do go to infinity when applied to a certain space or unbounded line divided into parts.[50] Now here there is a new difficulty. Is the last number of a series of this kind the last one that would be ascribed to the divisions of the unbounded line? It is not, otherwise there would also be a last number in the unbounded series. Thus if you say that in an unbounded series there exists no last finite number that can be written in, although there can exist an infinite one: I reply, not even this can exist if there is no last number.[51]

*Cantor*: I agree that in a series such as $^1/_2$, $^1/_4$, $^1/_8$, $^1/_{16}$, $^1/_{32}$, etc., there is no last number, not even an infinite one. If there were, it would be an actual infinitesimal. But there are no such things.

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[46] *Ibidem*.
[47] *Ibidem*. The importance of this proof has been noted by O. Bradley Bassler, who observes that here "Leibniz distinguishes between there being no greatest number and the number of finite numbers being infinite" (*op. cit*. p. 850).
[48] Leibniz, "On the Secrets of the Sublime", *LoC*, p. 53 (A VI.iii.477). Cf. also *De Quadratura Arithmetica*, p. 133: "Thus I call *unbounded* that in which no last point can be assumed, at least on one side; whereas we understand a quantity, whether bounded or unbounded, to be *infinite,* provided it is greater than any that can be assigned by us or designated by numbers." This is part of an addition to the Scholium that Leibniz subsequently cancelled.
[49] An example is given by Bassler in fn. 15 of his article, quoting the following sophism of Henry de Gand: "Infinities are finite. Proof: two are finite, three are finite, and so on to infinity; therefore infinities are finite." This assumes that there is an infinitieth finite number.
[50] In his "Leibniz on the Indefinite as Infinite" O. Bradley Bassler notes Leibniz's proof in "On the Secrets of the Sublime" that the number of finite numbers cannot be infinite, but then comments: "what he does not consider is the possibility of nonetheless admitting that there is an (indefinite) infinity of finite numbers" (p. 853, n. 12). But this passage from "Infinite Numbers", written only a few weeks later, shows that Leibniz did consider the possibility of an infinity of finite numbers, in the sense of there being more than can be numbered.
[51] Leibniz, "Infinite Numbers", *LoC*, p. 99 (A VI.iii.503).

*Leibniz*: I am glad we agree! I had occasion to argue this with one of the most eminent mathematicians of my time, Johann Bernoulli. He claimed that it was inconsistent of me to claim that any finite portion of matter is already actually divided up into an infinity of parts, and yet deny that any of these parts is infinitely small.[52] To this I replied that, even if we suppose a line to be divided into fractions of its length so that its $^1/_2$, $^1/_4$, $^1/_8$, $^1/_{16}$, $^1/_{32}$, etc. etc., are actually assigned, and all the terms of the series actually exist, it still does not follow that there also exists an infinitieth term.[53]

*Cantor*: Still, I hold that there can be infinite sequences such that there is an infinite number *after* the whole sequence. You don't know how many attempted proofs I have seen of the impossibility of infinite number that begin by presupposing that an actual infinite *must be unincreasable* in magnitude; an erroneous assumption that one finds propagated not only in the *old* philosophy following on from the Scholastics, but even in the *more recent* and, one can almost say, *most recent* philosophy.[54] On this basis it is easy to prove that actually infinite number leads to contradiction. I find articulated in several places in Gutberlet's *Works*, for example, the wholly untenable thesis that "in the concept of supposed infinite magnitudes lies the exclusion of all possibility of increase". This can be admitted only in the case of the Absolute Infinite.[55] By contrast, the transfinite, although conceived as definite and greater than every finite, shares with the finite the character of unbounded increasability.[56]

*Leibniz*: I do not understand how this increasability is compatible with the unvarying nature of the infinite you spoke of above when you claimed the potential infinite presupposed a definite, actually infinite set of values.

*Cantor*: By an Actual Infinite I understand a quantum that is *not variable*, but rather fixed and determined in all its parts, a genuine constant, but which at the same time exceeds in magnitude *every finite quantity* of the same kind.[57]

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52 Leibniz, Letter to Johann Bernoulli, GM.iii.535.
53 *Ibid.*, 536.
54 Cantor, "Mitteilungen" (1887-8), *GA* 404-5; cf. Hallett, p. 41.
55 *Ibid.*, *GA* 394; quoted from Hallett, p. 41. The reference is to Gutberlet's *Das Unendliche metaphys. und mathem. betrachtet* (Mainz, 1878).
56 *Ibidem.*
57 Cantor, "Mitteilungen" (1887-8), p. 401 ; cf. Hallett, p. 13.

*Leibniz*: But in what sense can the same infinite number be "fixed and determined in all its parts", a *genuine constant*, and at the same time be increasable?

*Cantor*: In exactly that sense that 3 can be increased to larger numbers by the addition of new unities.[58]

*Leibniz*: But when a unity is added to 3, it is no longer 3 but 4. So it is not 3 that is increasable, but the series of which it was supposed to be the last number. This I grant. Thus concerning the infinite series of numbers we were discussing above, the only other thing I would consider replying to that line of reasoning is that the number of terms is not always the last number of the series. That is, it is clear that even if finite numbers are increased to infinity, they never—unless eternity is finite, i.e. never—reach infinity. This consideration is extremely subtle.[59]

*Cantor*: But here we are *in* eternity, my dear Leibniz, so we should be able to find out! Seriously, though, I am amazed at the progress you had already made in this kind of reasoning in 1676. It is such a shame that you did not publish more of your reflections. For instance, in what you say here about the series (I) of positive real whole numbers 1, 2, 3, …, ν, …, you anticipate me by recognizing that the number [*Anzahl*] of so constituted numbers ν of the class (I) is infinite even though there is no greatest among them. For however contradictory it would be to speak of a greatest number of class (I), there is yet nothing objectionable in conceiving of a *new* number, which we will call ω, which is to be the expression of the fact that the whole domain (I) [of the positive real whole numbers] is given in its natural succession according to law.[60] One might also say that in distinguishing the number of terms of (I) from the last term of the series, you come very close to anticipating my distinction between *cardinal* and *ordinal* numbers[61]—with the exception that

[58] Cantor, *Nachlass* VI, pp. 47-48; quoted from Hallett, p. 41.
[59] Leibniz, "Infinite Numbers", *LoC*, p. 101 (A VI.iii.504).
[60] Cantor, "Grundlagen (1883)", *GA* 195.
[61] Commenting on the same passage, Samuel Levey has noted Leibniz's near anticipation of the cardinal/ordinal distinction: "In seeing his way clear to the fact that the number of terms in the series of natural numbers—the cardinality of the naturals—is not itself in the series, but rather lies outside it, Leibniz places himself well ahead of the majority of his peers on the topic. Further, taken at face value, his claim that 'the number of terms is not always the last number of the series' touches quite directly on the concept of cardinality, and conceives of a series' cardinality as a *number.* In the crucial case of the infinite series, the number of terms is the *cardinal number* "infinity" (waiving Cantor's distinctions between higher and lower transfinite cardinals), despite the fact there is no corresponding infinitieth element in the series." "Leibniz on Mathematics and the

what I call the first infinite ordinal number ω is not the last term in the series, but the first whole number which follows all the numbers ν of the series, that is to say, greater than each of the numbers ν.[62]

*Leibniz*: I admit that lack of time is no longer an obstacle for us. In fact I have taken advantage of our situation to study your "Characteristic of the Transfinite", if I may call it that. Cardinality, if I understand correctly, measures *how many* are in a given collection, without regard to their order; whereas when order is taken into account, many different infinite collections of the same cardinality have different ordinal numbers. But I regret to report that I am in no way persuaded of the existence of infinite numbers of either kind.

*Cantor*: Given time—or its complete absence—I am sure I can persuade you! The reason I am so confident is that you, like me, believe in the Principle of Plenitude.

*Leibniz*: You mean the principle that God would create as much as possible. This principle I also call the harmony of things: that is, that there exists the greatest quantity of essence possible. From which it follows that there is more reason for existing than not existing, and that all things would exist if that were possible.[63]

*Cantor*: Precisely! Now, in the transfinite there is a vastly greater abundance of forms and of *species numerorum* available, and in a certain sense stored up, than there is in the correspondingly small field of the unbounded finite. Consequently, these transfinite species were just as available for the bidding of the Creator and his absolutely inestimable will power as were the finite numbers.[64] Otherwise put: since God is of the highest perfection one can conclude that it is possible for Him to create a *transfinitum ordinatum*. Therefore, in virtue of His pure goodness and majesty, we can infer the necessity of the actually resulting creation of a *transfinitum*.[65]

*Leibniz*: But you still have not yet established that possibility.

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Actually Infinite Division of Matter," *The Philosophical Review*, 107, 1 (January 1998), 49-96: p. 84.

[62] Cantor, "Grundlagen (1883)", *GA* 195.

[63] Leibniz, "On the Secrets of the Sublime", *LoC*, p. 45 (A VI.iii.472).

[64] Cantor, "Mitteilungen" (1887-8), *GA* 404, fn; Hallett, p. 23.

[65] Cantor, *ibid.*, *GA* 400; Hallett, p. 23.

*Cantor*: I urge you to appreciate that these transfinite cardinal and ordinal numbers possess a definite mathematical uniformity, just as much as the finite ones do, a uniformity that is discoverable by men. *And all these particular modes of the transfinite have existed from eternity as ideas in the Divine Intellect.*[66] Such a transfinite imagined both *in concreto* and *in abstracto* is free from contradiction, and thus possible, and therefore just as much creatable by God as a finite form.[67]

*Leibniz*: It can be said to be possible that 1 = 2 if division by zero is allowed. This does not mean that it is something that God could instantiate in the world.

*Cantor*: Then what sense can be made of your claim that matter is actually infinitely divided if there is not an actually infinite collection of the parts that have been divided?

*Leibniz*: Exactly the sense that I was trying to explain before, when I spoke of the Scholastic distinction between the categorematic and the syncategorematic. You made light of this distinction, assimilating the latter to the potential infinite. But it seems to me the distinctions of the Scholastics are not entirely to be scorned. For I believe that there is an actual infinity of parts into which any piece of matter is actually (not merely potentially) divided, but that there is no totality or collection of all these parts. And this kind of actual infinite can be distinguished from the actual infinite understood categorematically using predicate calculus.

*Cantor*: How? I would like to see how this could be done.

*Leibniz*: I was on the point of showing you. Here, I'll write on this slate so you can see. To say of the prime numbers, for instance, that there is an actual infinity of them understood *syncategorematically*, is to say that no matter how large a number N one takes, there is a number of primes M, where M > N. Thus if *x* and *y* are numbers that can be assigned to count the primes, with *x* finite, then [*sound of writing*] $\forall x \exists y\ y > x$. But to assert their infinity *categorematically* would be to assert that there exists some one number of primes *y* which is greater than any finite number *x*, i.e. that [*sound of writing*] $\exists y \forall x\ y > x$. Thus the first way of expressing it does not commit you to infinite number, since *x* and *y* may both be finite. But the

[66] Cantor, Letter to Jeiler, 1895: *Nachlass* VII, p. 195; Hallett, p. 21.
[67] Cantor, Letter to Jeiler, 1895: *Nachlass* VII, p. 195; Hallett, p. 20. Cf. *GA* 396.

categorematic expression commits you to the existence of a number greater than all finite numbers.

*Cantor*: But if *x* and *y* may both be finite, in what sense does the first expression commit you to an actual infinite? This seems to me to be the potential infinite, beloved of the constructivists.

*Leibniz*: The primes are actually infinitely many in precisely the sense that if one assumes that their multiplicity is finite one may derive a contradiction, as did Euclid in his proof.

*Cantor*: But if the assumption that the number of primes is finite leads to a contradiction, why should one not conclude that their number is infinite, i.e. is an infinite number?

*Leibniz*: I grant you that there are actually infinitely many. But from Euclid's proof one cannot validly conclude that there is a categorematic infinity. For he begins his proof by supposing that there is a greatest prime, i.e. that there is some prime such that all primes are less than or equal to it. That is, if *x* and *y* are numbers that can be assigned to count the primes, and $Fx = x$ is finite, then there is an *x* such that any *y* different from *x* must be less than or equal to it: in symbols, $\exists x \forall y[Fx \,\&\, (y \neq x \rightarrow y \leq x)]$. Then he shows how to construct a prime greater than this, thus contradicting the initial supposition. (One simply forms the product of this supposed greatest prime with all the preceding primes and adds one; this new number is not divisible by any prime without remainder, and is therefore a prime number greater than the supposed greatest.)

*Cantor*: Granted.

*Leibniz*: But the negation of the original supposition is [*more sounds of writing*] $\neg\exists x \forall y[Fx \,\&\, (y \neq x \rightarrow y \leq x)]$. This is equivalent to $\forall x \exists y \neg[Fx \,\&\, (y \neq x \rightarrow y \leq x)]$, or $\forall x \exists y[Fx \rightarrow \neg(y \neq x \rightarrow y \leq x)]$, or $\forall x \exists y[Fx \rightarrow (y \neq x \,\&\, y > x)]$: for any finite number of primes, there is a number of primes different from and greater than this. But this is the syncategorematic infinite, and from this one cannot infer the categorematic infinite, i.e. that there is a number of primes different from and greater than any finite number: "$\forall x \exists y\ y > x$" *ergo* "$\exists y \forall x\ y > x$", *non valet*.[68]

*Cantor*: I think this is now usually called the *quantifier shift fallacy*.

*Leibniz*: I believe so. I once suspected the great Locke to be guilty of it when he reasoned: "Bare nothing cannot produce any real being. Whence it follows with mathematical evidence that something has existed from all eternity."[69] I find an ambiguity here. If by 'something has existed from all eternity' he means that *there has never been a time when nothing existed*, then I agree with it, and it does indeed follow from the previous propositions with mathematical evidence. But if there has always been something it does not follow that one particular thing has always been, i.e. that there is an eternal being.[70] Still, if Locke did commit this error, it was not because he was deficient in logic: we can all reason badly, no matter how eminent we may be as logicians. Even Bertrand Russell committed this fallacy, and it would be hard to find a more distinguished logician than him.

*Cantor*: I am surprised to hear that! Lord Russell was, next to Ernst Zermelo, perhaps the ablest champion of my theory of the transfinite, and his and Whitehead's *Principia Mathematica* was a pinnacle in the history of logic.

*Leibniz*: Yet in his *Principles of Mathematics* (1903) he wrote: "Of some kinds of magnitude (for example ratios, or distances in space and time), it appears true that [i] there is a magnitude greater than any given magnitude. That is, [ii] any magnitude being mentioned, another may be found greater than it."[71] Now 'That is' is a particle denoting logical equivalence; but although [ii] follows from [i], to assert that the converse is valid is to commit the quantifier shift fallacy. Russell conflates the syncategorematic with the categorematic infinite.[72]

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[68] I am indebted to my colleagues Nick Griffin, Gregory Moore and David Hitchcock for observing that the finiteness of *x* and its distinctness from *y* needed to be made explicit in this proof.
[69] Leibniz, *New Essays*, A VI 6, 435; quoted from Locke, Essay, Bk. IV, ch. x, §3.
[70] Leibniz, *New Essays*, A VI 6, 436. In terms of predicate logic, if $xTy$ = $x$ exists at time $y$, then *there has never been a time when nothing existed* is $\neg\exists y \forall x \neg xTy$, which, as Leibniz says is equivalent to $\forall y \exists x\ xTy$, *there has always been something*. But to infer from this that $\exists x \forall y\ xTy$, *one particular thing has always been*, is to commit the quantifier shift fallacy.
[71] Bertrand Russell, *Principles of Mathematics* (New York: W. W. Norton, 1903), p. 188.
[72] As has been pointed out to me by Wayne Myrvold and Dean Buckner, in the light of his subsequent discussion (188-89) my 'Leibniz' should probably have given Russell the benefit of

*Cantor*: I am grateful here for your indirection or tact in not accusing me of this fallacy. Actually, I do not say that the transfinitude of the primes (for example) directly follows from the falseness of the assumption of their finitude. Rather I prefer to say that, just as the actual infinity of monads is the foundation of extended matter, so the categorematic infinite is the foundation of the syncategorematic. There would not be an infinity of divisions of the continuum without an actually infinite collection of monads, each one underpinning a given part.

*Leibniz*: I know you put much stock in the fact that your transfinite numbers would be instantiated in reality ...

*Cantor*: Yes, that is so! I believe the suppression of the legitimate actual infinite can be viewed as a kind of shortsightedness that robs us of the possibility of seeing that, just as the actual infinite in its highest, absolute bearer has created and sustains us, so in its secondary transfinite form it occurs all around us and even inhabits our minds themselves.[73] So, for instance, the various number-classes (I), (II), (III) etc. are representatives of powers which are actually found in corporeal and intellectual nature.[74] For apart from pure mathematics (which in my view is nothing other than pure set theory),[75] there is also *applied set theory*, by which I understand what one cares to call *theory of nature*, or *cosmology*, to which belong all so-called natural sciences, those relating to both the inorganic and the organic world ...[76]

*Leibniz*: Yes, yes, but what if I were to demonstrate to you that the actually infinite division of matter, as I conceive it, while consistent with my actual infinite that is syncategorematically understood, is incompatible with there being a transfinite number of parts?

*Cantor*: Please give the demonstration, and I will comment when I see it.

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the doubt here: Russell likely intended (ii) as a more exact expression of what was expressed somewhat stiltedly in (i), rather than as an equivalent form.

[73] Cantor, "Standpunkte (1885)", *GA* 374-5; cf. Rudy Rucker's translation (*Infinity and the Mind*, p. 46).

[74] Cantor, "Grundlagen (1883)", , *GA* 181; cf. Hallett, p. 18, and the distinction between the two kinds of reality of numbers: *intrasubjective* or *immanent reality* and *transsubjective* or *transient reality*.

[75] Cantor, "Principien einer Theorie der Ordnungstypen" (1885); Grattan-Guinness, I, "An unpublished paper by Georg Cantor…", *Acta Math.* 124, 1970, 65-107: p. 84.

[76] *Ibid.,* p. 85.

*Leibniz*: According to my conception, the parts of matter are instituted and individuated by their differing motions. Each body or part of matter would be one moving with a motion in common. But this does not rule out various parts internal to that body having their own common motions, which effectively divide the body within. Accordingly I ask that I be allowed the following three premises:

1) Any part of matter (or *body*) is actually divided into further parts.

2) Each such body is the *aggregate* of the parts into which it is divided.

3) Each part of a given body is the result of a division of that body or of a part of that body.

*Cantor*: Go on.

*Leibniz*: From 1) and 2) it follows that every body is an aggregate of parts. Incidentally, if one accepts the further premise that

4) What is aggregated cannot be a substantial unity.

this immediately yields one of my chief doctrines, that matter, being an aggregate, cannot be substantial. But I will not pursue that further here.

To resume: we have the result that every body is an aggregate of parts, each of which is an aggregate of further parts, and so on to infinity. That is, by recursion,

5) Every body is actually infinitely divided.

*Cantor*: —in your syncategorematic sense.

*Leibniz*: Exactly. And from 1) and 3) it follows that

6) Any part of a given body must be reachable by repeated division of the original body.

This is the crux of the matter—each part of a body must be reachable by repeated division of the original body. It is either a part, a part of a part, or a part of a part of a part, and so on. This makes for a recursive connection between a body and any of its parts: Call $P_r$ any part reachable by $r$ divisions of the original body; then by 6)

7) For all $r > 1$, the part $P_r$ must have resulted from the division of a part $P_{r-1}$.

*Cantor*: Here it seems to me that in introducing this concept of "reachable by division" you are illicitly appealing to time. I must declare that in my opinion reliance on the concept of time or the intuition of time in the much more basic and general concept of the continuum is quite wrong.[77]

*Leibniz*: I am sorry if I gave the impression that this "reaching" a part by a division is a process occurring *in time*. Indeed, it cannot be, since each motion causing these divisions within matter is an instantaneous motion. It is just that in order for a piece of matter to be a part of the original body, it must be a result of a division of the body.

*Cantor*: Very well. But it still seems to me that if the body is actually infinitely divided at an instant, then all the infinite divisions simply are there. And if this is so, then there is a number of divisions $\omega$, greater than any finite number *r* of divisions. Here $\omega$ is the first ordinal number > *r* for all finite numbers *r*, i.e. the first transfinite ordinal. But you were going to prove that this is incompatible with actually infinite division?

*Leibniz*: Yes. Here is the demonstration, which I have adapted from one communicated to me by a scholar from Spain.[78] Suppose that a given body has an actual infinity of parts in your categorematic sense, i.e. suppose that there is a number of divisions $\omega$, greater than any finite number *r* of divisions. Let $P_\omega$ be a part resulting from the $\omega^{th}$ division ...

*Cantor*: Excuse me for interrupting, but here I must correct you. If there is a number of divisions $\omega$, this means that there will be a well-ordered sequence of divisions of order type $\omega$. But in that case there will be a number of divisions greater than any finite number, but no infinitieth division, or part $P_\omega$. More formally, the $\omega$-sequence of parts $\{P_1, P_2, P_3, ...\}$ does not issue in an $\omega^{th}$ part $P_\omega$: that would require a sequence of divisions of order type $\omega + 1$, namely $\{P_1, P_2, P_3, ..., P_\omega\}$. This relates to what we were discussing earlier, when I agreed with your answer to Bernoulli.

[77] Cantor (1932, pp. 191-2), quoted from the translation in Hallett, p. 15.
[78] I owe this form of argument to Antonio Leon, who used it in his article of his published on the internet which he generously brought to my attention: "On the infiniteness of the set of natural numbers", http://www.terra.es/personal3/eubulides/infinit.html (now expired) He interpreted it to show that the natural numbers cannot be infinite in Cantor's sense.

*Leibniz*: I take your point. Still, if I understand you correctly, the transfinite really *begins* with $\omega$: the first transfinite division would be the first after all the finite ones, assuming there is such a thing.

*Cantor*: This is true in the sense that $\omega$ is the first ordinal number after all the finite ordinal numbers. That it exists follows from the *second principle of generation* of whole real numbers, which I define more exactly as follows: if there is some determinate succession of defined whole real numbers, among which there exists no greatest, on the basis of this second principle of generation a new number is obtained which is regarded as the *limit* of those numbers, i.e. is defined as the next greater number than all of them.[79] Thus $\omega$ is the first ordinal greater than all the natural numbers in their natural sequence.

*Leibniz*: This is what I had in mind by referring to a "transfinite division". So long as we are considering a part $P_r$ with *r* finite, we have not yet attained your realm of the transfinite. But if we suppose that there is a part $P_t$ occurring as a result of a transfinite division, so that *t* is the first ordinal greater than every finite *r*, then $t = \omega$.

Now I adapt my Spanish correspondent's proof as follows: suppose there is a division issuing in an $\omega^{th}$ part $P_\omega$, where $\omega$ is the least ordinal number greater than any finite number *r*. The point is that by 7) above, $P_\omega$ must have resulted from the division of a part $P_{\omega - 1}$. But $\omega - 1$ cannot be infinite, otherwise $\omega$ would not be the first ordinal number > *r* for all *r*, contrary to your supposition. But neither can it be finite, since one cannot obtain an actually infinite division by one more division of a finite aggregate of parts. Therefore there cannot be a transfinite division, i.e. one issuing in an $\omega^{th}$ part $P_\omega$.

*Cantor*: It seems to me that all your friend has proved by this kind of argument is that one cannot obtain the transfinite through ordinary recursion from the finite.

*Leibniz*: I think he has shown much more. For recursive connectibility is a crucial property of numbers for many mathematical proofs, and the demonstration shows that it is lacking in your transfinite ordinals.

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[79] Cantor, "Grundlagen (1883)", *GA* 196.

*Cantor*: Here it seems you are unaware of my generalization of ordinary, Fermatian recursion. In standard proofs by mathematical induction, one standardly shows first that *F* holds for some first natural number, and then that if *F* holds for *n*, it also holds for *n* + 1, and concludes that it must hold for any natural number. But there is a generalization of this which we may call by the happy name "transfinite recursion". Here one first shows that *F* holds for 0, and then that, for any ordinal number *k*, if *F* holds for all ordinals less than *k*, it also holds for *k*, and concludes that it must hold for all ordinals. It is readily seen that this is applicable to all ordinals, transfinite as well as finite, so that Fermatian recursion is but a special case.[80]

*Leibniz*: That is very ingenious, and I thank you for enlightening me. But note that this generalized type of recursion only applies to the transfinite on the hypothesis that transfinite numbers exist. In this it differs from standard recursion, which in a sense generates the applicability of *F* to the successor numbers. Similarly, as I understand the notion of a 'part', it must be generated by a division, and this requires standard recursion. But a "transfinite part", if I may call it that, is separated from the whole of which it is putatively a part by a kind of rift. This rift between the finite and the transfinite is, in my view, a mark of the inapplicability of the transfinite to reality, at least in this case of infinite division.

*Cantor*: I cannot agree, for it remains the case that infinite division as described by you can be represented in my theory by an actually infinite $\omega$-sequence of parts $\{P_1, P_2, P_3, \ldots\}$ whose power or cardinality is $\aleph_0$. Indeed it must be so represented if the division is completed. If, however, the division is incomplete, then the infinity in question is merely potential. It is one whose parts are variable quantities which are decreasing towards any arbitrary smallness, but which always remain finite. I call this infinite the *improper infinite*.[81] Thus I still am not convinced that your syncategorematic but actual infinite is not a contradiction in terms. Are you not presupposing the categorematic actual infinite when you use the universal quantifier?

[80] This is a somewhat simplified version of transfinite induction (apparently not Cantor's own term), which I owe to Wayne Myrvold. I am also in his debt for alerting me to the theory, and to its significance for this discussion. For Cantor's rigorous formulation, see his "Beiträge (1887-8)", *GA* 336-339.
[81] Cantor, "Grundlagen" (1883), *GA* 165.

*Leibniz*: I allow that this is how it is usually understood. But one may interpret it, as I have here, to mean "For each *x* ..." rather than "For all *x* ..." In this way one sees the link with another useful distinction from the Schools, namely that between the distributive and the collective use of a term like "every". For when we say "every man is an animal" or "all men are animals", the acceptation is distributive: if you take that man (Titius) or this man (Caius), you will discover him to be an animal.[82] According to this distinction, then, there is an actual infinite in the mode of a distributive whole, not in that of a collective whole. Thus something can be enunciated concerning all numbers, but not collectively.[83]

*Cantor*: I disagree. If I say "The apostles are 12", the number 12 does not hold of the apostles individually.[84] Any number applied to a multiplicity must be understood collectively: there is a 1-1 correspondence between this multiplicity *M* (collectively) and any other multiplicity of 12 elements. The cardinality of a set is what you get by abstracting away from all other properties. If with a given set *M*, which consists of determinate, well-differentiated concrete things or abstract concepts which are called the elements of the set, we abstract away not only from the particular character of the elements but also from the order in which they are given, there arises in us a determinate general concept that I call the *power* of *M* or the *cardinal number* of the set *M* in question.[85] Thus, for example, the set of colours of the rainbow (red, orange, yellow, green, blue, indigo, violet) and the set of pitches in the octave (C, D, E, F, G, A, B) are equivalent sets and both stand under the same cardinal number *seven*.[86]

*Leibniz:* I allow that *numbers* must be understood collectively when applied to finite multiplicities. But the universal quantifier is not always used collectively, even when the collection is finite. For instance, when one says "All the children in my class

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[82] Leibniz, Preface to an Edition of Nizolius (1670); quoted from *Philosophical Papers and Letters*, ed. and trans. Leroy Loemker (Dordrecht: D. Reidel, 1969), p. 129. Leibniz goes on to say that "If ... 'every man', or 'all men', is taken as a collective whole, and the same as the whole genus man, then an absurd expression will result ...: 'the whole genus man is an animal'." Of course, with the replacement of the logic of class inclusion by predicate logic, the criticism no longer applies in this form. On the one hand, Leibniz's analysis of an A-statement as a (possibly infinite) conjunction of conditionals is vindicated; on the other, one may interpret an infinite domain of the universal quantifier as an actually infinite set (as do Platonist mathematicians), or as a potentially infinite set (as do constructivists), or as an actual infinite understood distributively (as does Leibniz).
[83] GP II 315
[84] I am indebted to Massimo Mugnai for this objection (private correspondence).
[85] Cantor, "Mitteilungen" (1887-8), *GA* 411.
[86] *Ibid*, 412.

weigh less than 40 kg," one most likely means that each child does, not that all the children put together do. Nothing is being said about the collection of children in the room: the quantifier is being used distributively. It is similar when the universal quantifier is used with an infinite multiplicity, like the natural numbers. Here something can be enunciated concerning all numbers, but not collectively. Thus it can be said that to every even number there is a corresponding odd number, and vice versa; but it cannot accurately be said that the multiplicity of even numbers is therefore equal to that of odd ones. [87]

*Cantor*: I grant you that we cannot infer the existence of infinite collections from a proof that they are not finite, since that would involve the fallacy we noted above. But the existence and equality of infinite collections can still be posited. Indeed, I believe that this is how we must proceed. Mathematics is completely free in its development, and bound only by the self-evident consideration that its concepts must not only be free from contradiction but also stand in ordered relations, fixed through definitions, to previously formed concepts that are already present and tested.[88]

*Leibniz*: I agree with this. But I do not believe these very concepts, the number of all even numbers, and the number of all numbers, are consistent with "concepts already present and tested". For the number of all numbers implies a contradiction, which I show thus: To any number there is a corresponding number equal to its double. Therefore the number of all numbers is not greater than the number of even numbers, i.e. the whole is not greater than its part.[89] Galileo showed the same thing using the number of all squares, and avoided the contradiction by denying that "equal to" and "lesser than" can be applied to the infinite. Here the axiom on which the proof depends, *Totus est majus sua parte*, is not only "present and tested", but something I believe I was the first to prove.

*Cantor*: The old and oft-repeated proposition "*Totum est majus sua parte*" may be applied without proof only in the case of *entities* that are based upon whole and part; *then* and *only then* is it an undeniable consequence of the concepts "totum" and "pars". Unfortunately, however, this "axiom" is used innumerably often without any basis

[87] GP II 315
[88] Cantor, "Grundlagen" (1883), *GA* 182; cf. Hallett, p. 16.
[89] Leibniz, GP I 338.

and in neglect of the necessary distinction between "reality" and "quantity", on the one hand, and "number" and "set", on the other, precisely in the sense in which it is *generally false*.[90] Your example may help to explain. Let *M* be the totality (ν) of all finite numbers ν, and *M′* the totality (2ν) of all even numbers 2ν. Here it is undeniably correct that *M* is *richer* in its entity, than *M′*; *M* contains not only the even numbers, of which *M′* consists, but also the odd numbers *M″*. On the other hand it is just as unconditionally correct that *the same* cardinal number belongs to both the sets *M* and *M′*. Both of these are certain, and neither stands in the way of the other if one heeds the distinction between *reality* and *number*.[91]

*Leibniz*: This puts me in mind of an objection raised by a distinguished metaphysician of more recent times, who accused me of equivocation in deriving a contradiction from cases such as this.[92] For, he claimed, the sense in which the number of even numbers can be said to be less than the number of numbers is according to a criterion of equality that "if B is a proper subset of A, then there is a greater number of elements in A than in B"; whereas the sense in which they are equal is that their elements can be put into one-one correspondence. Once this difference in the meaning of equality is taken into account, he alleged, "there is no contradiction, as Leibniz [had] supposed [there to be]".[93]

*Cantor*: That is a little different from my objection. But I take it you have a reply?

*Leibniz*: Yes, and I think it also answers your criticism. I deny that I am taking equality in more than one sense. I certainly agree with you that any two aggregates that form wholes and whose elements can be put into 1-1 correspondence, are equal. But I do not believe one can determine which of two aggregates is greater or equal unless they are both wholes. This is the basis of a proof I once proposed of the

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[90] Cantor, "Mitteilungen", *GA* 416-417.
[91] Cantor, "Mitteilungen", *GA* 412.
[92] Benardete, José A., *Infinity: An Essay in Metaphysics*, Oxford: Clarendon Press, 1964; pp. 44-48. I am indebted to Mark van Atten for drawing this passage to my attention (although I was unable to convince him that Leibniz is not guilty of equivocation), and to Adam Harmer for his insights into how Benardete's criticism is misplaced (both private communications).
[93] Benardete, *Infinity*, 47. Here I have switched A and B for convenience. Benardete also distinguished a third criterion of equality, "If all the elements of [B] can be put into 1-1 correspondence with a proper subset of [A], then [A] contains a greater number of elements than [B]." If 1-1 correspondence is taken to define equality when applied to sets of numbers taken as wholes, then the latter constitutes an application of Leibniz's definition of "less than" in his proof of the part-whole axiom. But it is not a separate criterion of equality.

Euclidean part-whole axiom by means of the following syllogism, which depends on defining "less than" in terms of "[proper] part" of:

Whatever is equal to a part of *A* is less than *A*, by definition.
*B* is equal to a part of *A* (namely, to *B*), by hypothesis.
Therefore *B* is less than *A*.

This is the sense in which we can say that the aggregate of even numbers is less than the aggregate of natural numbers: B, the aggregate of all even numbers, is equal to itself (by any criterion equality must be reflexive), and is a proper part of A, in that A is the union of B with the odd numbers. One cannot generate any contradiction unless one assumes that A and B are wholes, one of which can be part of the other. For it is only on this basis that one can infer that one has fewer elements than the other.

*Cantor*: I completely concur! Only I deny that infinite sets (or aggregates, as you call them) *are* entities that are based upon whole and part. In this way I avoid the contradiction...

*Leibniz*: ... whereas I believe it to be no less true in the infinite than in the finite that the part is less than the whole.[94] If one accepts (as you do not) the supposition that the infinite aggregate of natural numbers forms a whole to which this axiom or the definition from which I prove it applies, then this entails a contradiction. I therefore reject the supposition: that is, I concede the infinite multiplicity of terms, but this multiplicity does not constitute a number or a single whole. It signifies only that there are more terms than can be designated by a number. [95] Consequently I do not concede your claim that the multiplicity of even numbers is equal to the multiplicity of natural numbers or of odd ones if one interprets this equality as sameness of number.[96]

*Cantor*: Still, I think you misunderstood my remark about relations to concepts already present and tested. If every new concept had to agree with all the old definitions, there would be no new mathematics. What I mean is that if one can define a new

[94] Leibniz, *Pacidius Philalethes,* A VI iii 551; LoC .
[95] Leibniz, Letter to Bernoulli, GM III 575.
[96] GP II 315.

concept—such as, here, equality of two infinite sets—in such a way that it is internally consistent, and stands in a determinate relation to old ones, then it should be allowed; and, because of the principle of plenitude, every consistent concept will be instantiated. In particular, in introducing new numbers [mathematics] is only obliged to give them definitions through which [*Cantor coughs*] such a relation is conferred on them to the old numbers that they are determinately distinguishable from one another in given cases.[97] So, not only do I believe that the idea of an infinite number is consistent; I believe also that there are different "sizes" of what I call the transfinite. For instance, although the actually infinite set (ν) of all positive finite whole numbers ν is equivalent to the set (μ/ν) of all positive rational numbers μ/ν, where μ and ν are prime relative to one another, the set of *all real* number magnitudes is *not* equivalent to the set (ν).[98] Thus, whereas the number of all rational numbers is the same as the number of natural numbers, the number of all real numbers is greater than either of these numbers, as I think I have demonstrated.

*Leibniz*: Yes, I have heard of these celebrated proofs of yours, and am eager to discuss them with you. I think it will pay us to go through them most carefully. I believe that you establish the first claim by pairing off the rational numbers with the natural numbers in such a way that one can be sure that every rational is counted?

*Cantor*: Precisely! One simply makes an infinite array as follows:

| | | | | | |
|---|---|---|---|---|---|
| 1/1 | 2/1 | 3/1 | 4/1 | 5/1 | ... |
| 1/2 | 2/2 | 3/2 | 4/2 | 5/2 | ... |
| 1/3 | 2/3 | 3/3 | 4/3 | 5/3 | ... |
| 1/4 | 2/4 | 3/4 | 4/4 | 5/4 | ... |
| 1/5 | 2/5 | 3/5 | 4/5 | 5/5 | ... |
| ... | ... | ... | ... | ... | ... |

Now if one goes through this list by beginning at the top left, moving to the right to 2/1 and then cutting diagonally down to 1/2, then down to 1/3, and then cutting diagonally back to through 2/2 to 3/1, across to 4/1, and then cutting diagonally

[97] Cantor, "Grundlagen" (1883), *GA* 182; an inessential elision where Cantor coughs.
[98] Cantor, "Mitteilungen", *GA* 412,

down to 1/4, proceeding down to 1/5 and then back up through the next diagonal 3/3 to 5/1, and so forth, one can proceed through the whole list without missing out a single rational number. If it be objected that 2/2, 3/3 etc., or 1/2, 2/4, 3/6 etc., are different numerals for the same number, we can simply agree to cross out all recurrences of the same number before counting them.

*Leibniz*: I agree that this shows that the rational numbers are countable, or *denumerable*, as you say, in the sense that they can be paired off with the natural numbers that we use for counting. But this does not prove that they form a *totality* or that there is a *collection* of them.[99]

*Cantor*: But you will grant that I may define terms as I choose, provided no contradiction follows?

*Leibniz*: Of course. If one denies this, science becomes impossible.

*Cantor*: I am glad to see that you are not to be counted among those who reject the law of excluded middle, *tertium non datur*. Suppose, then, that I define the power of a collection or set, whether finite or not, as follows: We call two sets *M* and *N* "equivalent", and denote this by $M \approx N$ or $N \approx M$, if it is possible to set them in such a relation to one another that every element of one of them corresponds to one and only one element of the other. Two sets *M* and *N* have the same cardinal number if and only if they are equivalent.[100] Thus the previous proofs establish that the natural numbers, the even numbers and the rational numbers all have the same power or cardinal number. Now by abstraction we can say that all sets with the same cardinality have the same number of elements. That is, abstracting from the nature or specific character of the elements, as well as from the order in which they are given, all sets of the same cardinality have the same *cardinal number*. The cardinality of the set of natural numbers I define to be $\aleph_0$.

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99 Dean Buckner (in private correspondence) has drawn my attention to a passage from Wittgenstein's *Philosophical Remarks* in which he makes a similar point: "§ 141. Does the relation m = 2n correlate the class of all numbers with one of its subclasses? No. It correlates any arbitrary number with another, and in that way we arrive at infinitely many pairs of classes, of which one is correlated with the other, but which are never related as class and subclass. Neither is this infinite process itself, in some sense or other, such a pair of classes."

100 Cantor, "Beiträge" (1897), *GA* 283. Cf. "Grundlagen" (1883), *GA* 167: "To every well-defined set there belongs a determinate power, whereby two sets are ascribed the same power if they can be mutually and univocally ordered one to another element for element."

*Leibniz*: You may define things so. But you have not proved the possibility of your definitions, nor that $\aleph_0$ can properly be regarded as a number.

*Cantor*: I think you are working with too restricted a concept of number. No one has established the possibility of irrational numbers, but we all think these exist; and I believe that one can say unconditionally: the transfinite numbers *stand or fall* with the finite irrational numbers; they are like each other in their innermost being: for the former, like the latter, are determinate, delimited forms or modifications (αφωρισμένα) of the actual infinite.[101] Indeed, the provision of a foundation for the theory of irrational quantities cannot be effected without the use of the actual infinite in some form.[102] I assume from your silence that I may proceed. Suppose now that I take all the real numbers between 0 and 1. Now suppose we have some rule that enables us to compile an exhaustive list of these numbers ...

*Leibniz*: But already I cannot accept this, for it is precisely the character of the infinite that it cannot be exhausted.

*Cantor*: I grant this without reservation. It is indeed precisely the character of the transfinite that it can always be increased. But what I meant is illustrated by the example of the rational numbers. There we had a rule that allowed us to list all the rational numbers—in principle, or given all eternity (which, you will agree, we have at our disposal). The rule allows us to say that no rational will be missing from the list. I call such an enumerable collection of elements a *denumerable set*: each element has a corresponding natural number, and for each natural number there is a corresponding element. There is, for example, an $n^{\text{th}}$ prime number for any natural number $n$.

*Leibniz*: I have already agreed that the rational numbers can be enumerated in this way, but not that they form a collection.

*Cantor*: Assume that it is possible to form such a collection—for how else would you prove the notion contradictory? Now suppose that the real numbers form such a denumerable set. Suppose, for example, that we can enumerate the totality of all

[101] Cantor, "Mitteilungen" (1897-8), *GA* 395-396; cf. Shaughan Lavine, *Understanding the Infinite*, (Cambridge: Harvard University Press, 1994), p. 93; cf. Hallett, p. 80.
[102] Cantor, *Nachlass* VI, p. 64; quoted from Hallett, p. 26

the real numbers within any interval (α … β) as follows. Let there be any two symbols *m* and *w* that are distinct from one another. Now consider a domain [*Inbegriff*] *M* of elements $E = (x_1, x_2, \ldots, x_\nu, \ldots)$, which depend on infinitely many co-ordinates $x_1, x_2, \ldots, x_\nu, \ldots$, such that each of these co-ordinates is either *m* or *w*.[103] (Indeed, if we take *m* and *w* as 0 and 1, each of these elements *E* can be regarded as a numeral expressing the real number as a binary sequence. I am sure this would be acceptable to you in the light of your pioneering work on binary numbers.) Now *M* is the totality of all elements *E*. I claim that such a set *M* is not of the power of the sequence 1, 2, 3, …, ν, … . That is shown by the following theorem:

> If $S = E_1, E_2, \ldots, E_\nu \ldots$ is any simply infinite sequence of elements of the set *M*, then there is always an element $E_0$ that corresponds to no $E_\nu$.

To prove this, let

$$E_1 = (a_{11}, a_{12}, \ldots a_{1\nu}, \ldots),$$
$$E_2 = (a_{21}, a_{22}, \ldots a_{2\nu}, \ldots),$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$
$$E_\mu = (a_{\mu 1}, a_{\mu 2}, \ldots a_{\mu\nu}, \ldots),$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$

Here the $a_{\mu\nu}$ are *m* or *w* in a definite manner. Produce now a sequence

$$b_1, b_2, \ldots b_\nu, \ldots,$$

so defined that $b_\nu$ is different from $a_{\nu\nu}$ but is also either *m* or *w* (i.e., 0 or 1). Thus if $a_{\nu\nu} = 0$, then $b_\nu = 1$, and if $a_{\nu\nu} = 1$, then $b_\nu = 0$. If we now consider the element

$$E_0 = (b_1, b_2, b_3, \ldots),$$

of *M* (which element we may call the *antidiagonal* of *S*), we see at once that the equality

$$E_0 = E_\mu$$

can be satisfied for no whole number value for μ. Otherwise, for the μ in question and for all whole number values of ν, $b_\nu = a_{\mu\nu}$, and therefore in particular $b_\mu = a_{\mu\mu}$ would hold, which is ruled out by the definition of $b_\nu$. It follows immediately from this theorem that the totality of elements of *M* cannot be brought into the form of a sequence $E_1, E_2, \ldots, E_\nu \ldots$ ; we would otherwise be faced with the contradiction that a

[103] Cantor, "Über eine elementare Frage der Mannigfaltigkeitslehre" (1890-91), *GA* 278; cf. Lavine, *Understanding the Infinite*, p. 99.

thing $E_0$ both and is not an element of $M$.[104] Thus we have the result that *M*, the totality of all the elements *E*, has a power (or cardinality) greater than the power of $\mathcal{N}$, the number sequence 1, 2, 3, …, ν …. That is, the number of real numbers $\Re$ between, say, 0 and 1 is greater than can be enumerated, even using the infinity of natural numbers: $\mathcal{C}(\Re) > \mathcal{C}(\mathcal{N})$, or, $\mathcal{C}(\Re) > \aleph_0$.

*Leibniz*: This excellent demonstration does indeed prove that there is a contradiction in assuming that the number of all real numbers between 0 and 1 (or any other interval) is a denumerable totality. But to say that the number of real numbers is greater than the number of natural numbers is to say that both sets can be treated as consistent wholes such that one is greater than the other. However, as I already argued, Galileo showed long ago that if one treats infinite sets as wholes, then they cannot be compared as to greater and lesser.

*Cantor*: Yes, yes, I know the demonstration in his *Due Nuove Scienze*. Since to each (natural) number there is a corresponding square, and vice versa, they are equal. But there are natural numbers that are not squares, e.g. 2, 3, 5, 7, etc. So the number of natural numbers is greater than the number of squares. But you are forgetting that I have abandoned the definitions characteristic of finite numbers, and have chosen (after Dedekind's example) to call sets equal if they can be set in 1-1 correspondence, notwithstanding Bolzano's observation that an infinite set is one that can be put into such correspondence with its proper subset.

*Leibniz*: I had not forgotten. But by what right can you say that $\mathcal{C}(\Re) > \mathcal{C}(\mathcal{N})$ just because there are real numbers that are left out of the enumeration corresponding to natural numbers, when you have denied that one can infer that $\mathcal{C}(\mathcal{N}) > \mathcal{C}(\mathcal{S})$ on the grounds that there are natural numbers that are not squares?

*Cantor*: The difference is that if we drop the Euclidean axiom of wholes for infinite sets, namely that the whole (here, the set) is greater than the part (here, the proper subset), then equality of infinite sets is decided solely on the basis of 1-1 correspondence. In the case of the squares, the assumption of a 1-1

[104] Cantor, *ibid.*, *GA* 278-279; Lavine, pp. 99-102.

correspondence with the natural numbers does not lead to a contradiction. But in the case of real numbers, it does.

*Leibniz*: But what can one conclude from the contradiction if one has no assumption about wholes and parts? Surely you have to assume that the set $S = \{E_1, E_2, \ldots, E_\nu \ldots\}$ lists *all the real numbers*?

*Cantor*: Yes, I believe you are right. The proof assumes that this set numbers all the elements of $\mathfrak{R}$ between 0 and 1, that they form a denumerable totality. Otherwise, if it were not denumerably complete, $E_0$ might be one of the elements that is missing from it.

*Leibniz*: Allow me to make this assumption explicit, in a manner suggested to me by my correspondent from Spain. (I shall use your terminology, and speak of infinite collections as sets [*Menge*].) We may call a subset $T$ of a given infinite set $M$ a *complete enumeration* of $M$ iff (i) $T$ is a denumerable subset of $M$, i.e. its elements can be put into 1-1 correspondence with those of $N$ (*denumerabilty*); and (ii) no element of $M$ is left out of $T$ (*completeness*). Then we can agree to call $M$ a *denumerable totality* if and only if there exists a complete enumeration of $M$.

*Cantor*: Fine. In my argument I assumed that $T$ was such a complete enumeration of all the elements of $\mathfrak{R}$ between 0 and 1, which would be your set $M$. Now if one forms the antidiagonal $t_0$, then by construction $t_0 \notin T$, but $t_0 \in M$. Therefore there are more elements in $M$ than are contained in the complete enumeration $T$, i.e. non-denumerably many.

*Leibniz*: But it seems to me that $T$ is not a complete enumeration by this definition, since $t_0$ is an element of $M$ left out of $T$, so that condition (ii) above fails. Nor may we avoid this by including the antidiagonal in a new enumeration, forming the subset $T'$ of $M$ which contains all the elements $t_i$ of $T$ and also its antidiagonal $t_0$, so that $T' = T \cup \{t_0\}$. For $T'$ is a denumerable subset of $M$, since its elements can be put into 1-1 correspondence with $N$: $t'_1 = t_0$, $t'_n = t_{n-1}$, n > 1. (I believe Professor Hilbert used an argument like this concerning a certain infinite hotel).

*Cantor*: Obviously I can see why this still does not give us a complete enumeration: *T′* itself may be diagonalized, so that a new element $t'_0$ of *M* exists that is not included in the enumeration. Therefore *T′* is not itself a complete enumeration of *M*..

*Leibniz*: Exactly! And this argument can be reiterated to infinity. So we have proved that *there are no complete enumerations of M*. Yet, on the other hand, *each such subset T is a denumerable one*. Thus *M* is enumerable by infinitely many denumerable subsets, although none of these is a complete enumeration. So your conclusion should not have been that *M* is nondenumerable, but that it has no complete enumeration, and is therefore *not a denumerable totality*. This seems to me to indicate what I have been urging all along: you cannot treat an infinite multiplicity as a complete collection of elements.

*Cantor*: I am not persuaded. For I do not see why, once I have made the assumption that *T* lists all the real numbers, any further assumption is needed about completeness: if *all* of them are included, this *means* that none are left out. So I do not believe that you have shown any contradiction in the notion of an infinite collection. Moreover, I do not have to make a separate assumption that the real numbers form a set, provided only that you allow me that the natural numbers constitute a totality or set *N*, together with the assumption I made above that if *L* is a set, then so is the domain of all functions from *L* into an arbitrary fixed pair, such as 0 and 1.[105] Indeed, the above proof seems remarkable not only because of its great simplicity, but especially also because the principle employed in it is extendible without further ado to the general theorem that the powers of well-defined multiplicities have no maximum, or what is the same, that for any multiplicity *L* another *M* can be placed beside it that is of greater power than *L*.[106]

[105] Lavine, (*Understanding the Infinite*, p. 94) notes that although this proof of Cantor's is superior to his previous ones in that it "is independent of any notion of limit or of transfinite number", it "does, however, require a new set-existence principle: if *L* is a set, then so is the domain of all functions from *L* into an arbitrarily fixed pair" (94). As he observes, this is "equivalent to the Power Set Axiom in common use today: the subsets of a set form a set" (95), since any one function from a set *L* to a fixed pair is fully determined by the subset of *L* that is taken by the function to a fixed member of the pair. So the set of all such subsets is canonically identifiable with the domain of all such functions from *L* into some fixed pair.
[106] Cantor, "Eine elementare Frage" (1890-91), *GA* 279; cf. Lavine 101.

*Leibniz*: But, to take up my previous thread again, how can you claim that one set or whole is greater than another without some principle to the effect that the whole is greater than the part?

*Cantor*: The basis for the claim that any transfinite set *M* is greater than another *L* is that *L* is equal to a subset of *M*, and is not equal to *M*. Here equality of two sets is to be understood in terms of their being of the *same power*, that is, (as I explained before), *L* is equal to *M* if a one-one correspondence between their respective elements is possible.

*Leibniz*: If I may be so bold, I suggest this amounts to a replacement of the classical definition of *part*, but *not* a rejection of the Euclidean axiom that the whole is greater than the part. Going back to my proof of the Euclidean axiom,

Whatever is equal to a part of *A* is less than *A*, by definition
*B* is equal to a part of *A* (namely, to *B*), by hypothesis.
Therefore *B* is less than *A*.

Here 'whole', as applied to collections or sets, is to be understood as the *set A*, and 'part' as one of its *proper subsets B*. As I understand your proposal, you wish to replace the major proposition of the syllogism with the following definition:

*Whatever is equal to a subset of A and is not equal to A, is less than A,*

where two sets are considered equal whenever a one-one correspondence between their respective elements is possible. But this is simply to define a *(proper) part* of a transfinite set A as *a subset of A that cannot be put into 1-1 correspondence with A.* Indeed, this definition will also be valid for finite sets, since a proper subset of a finite set A is a subset of A that cannot be put into a one-one correspondence with it. I submit, therefore, that you do after all accept the Euclidean axiom that the whole is greater than the part, with *(proper) part* redefined to agree with your generalized definition of equality of sets.

*Cantor*: That is very ingenious. But it seems to me much more natural to regard any subset as a part, but to abandon the Euclidean axiom. Once I have defined 'less than' as above, I really have no need for any notion of 'whole' and 'part' of infinite sets or collections.[107]

*Leibniz*: But you will grant, I think, that some infinite multiplicities cannot consistently be regarded as wholes in the sense of completed collections?

*Cantor*: You are correct. If we start from the concept of a determinate multiplicity (a system, a domain [*Inbegriff*]) of things, it is necessary, as I discovered, to distinguish between two kinds of multiplicities (I always mean *determinate* multiplicities). For a multiplicity can be so constituted that the assumption of a "being together" of *all* its elements leads to a contradiction, so that it is impossible to construe the multiplicity as a unity, as "one finished thing".[108]

*Leibniz*: This is what I was urging you to concede just now about the denumerable totalities on which your diagonal argument can be seen as based.

*Cantor*: I do not accept that those were examples of such *absolutely infinite* or *inconsistent multiplicities*, as I call them. The latter have to do with the absolute, and are not susceptible to number. As one can easily convince oneself, "the domain of everything thinkable", for example, is such a multiplicity.[109]

*Leibniz*: I think I see the reason for your calling this inconsistent. The idea that this domain includes the collection of *all* thinkable things is immediately inconsistent with the fact that we can think of the domain itself, and this is necessarily something different from its elements.[110]

*Cantor*: Yes, this relates to an argument given in 1872 by my friend Richard Dedekind for the actual infinity of possible rational thoughts. It depends on the idea that a rational thought cannot represent itself. Then, if *s* is some such possible rational

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[107] It should be noted that Cantor himself defines *part* as follows: "'We call 'Part' or 'Subset' [*Teilmenge*] of a set *M* every *other* set $M_1$ whose elements are alike elements of *M*. " ("Beiträge" (1897), *GA* 282). For 'Cantor''s response here, I am indebted to Brad Bassler.

[108] Cantor, Letter to Dedekind, 28 July 1899; *GA* 443. Hallett notes (p. 166) that Grattan-Guinness has shown that this "letter" is really an amalgam by Zermelo of several letters written at different times.

[109] *Ibidem.*

[110] Cf. Rudy Rucker, 51: "Again, the reason that it would be a contradiction if the collection of all rational thoughts were a rational thought *T* is that then *T* would be a member of itself, violating the rationality of *T* (where "rational" means non-self-representative)."

(i.e. non-self-representative) thought, then so is "s is a possible thought", and so on to infinity.[111] Bernard Bolzano had given a similar argument regarding truths.[112]

*Leibniz*: But this seems an unfortunate example, since it seems it would also make Bolzano's "set of all absolute propositions and truths" an inconsistent totality: yet that was precisely the example he gave to show the existence of an absolutely infinite set.[113]

*Cantor*: This may be unfortunate, but it must be conceded. Similarly, it would seem to rule out the young Wittgenstein's view that "the world is the totality of facts"; for if that were a non-self-representative fact, it should have been included in itself, which is a contradiction. Allow me to spell out the argument more formally:

It depends on what we may call the *principle of the genetic formation of sets*, according to which a set is formed in two distinct stages: (i) some multiplicity of elements is given, and (ii) if some of these can be consistently combined into a unity, then this unity is a set.[114] Now let us call *T* the totality of thinkable things. If *T* is 'one finished thing', it must contain *all* thinkable things: that is, we must assume that all thinkable things being given, they may be collected together in thought to form *T*. But *T* itself is a thinkable thing not contained in the original totality. And if now we include *T* in the totality, another totality *T′* could be formed that would include *T*; but clearly the same argument could be applied to *T′*. Thus there is no escape: either all thinkable things do not form a totality, or *T* itself is not a thinkable thing.

*Leibniz*: And if I am not mistaken, Russell found in a paper of Burali-Forti a similar argument against the notion that there could be a set of all ordinals—one that you had already anticipated.

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111 Richard Dedekind, *Essays on the Theory of Numbers* (New York: Dover Publications, 1963), p. 64; originally 1872; cfr. Rucker, pp. 50, 335.

112 "The class of all true propositions is easily seen to be infinite. For if we fix our attention upon any truth taken at random …, and label it *A*, we find that the proposition conveyed by the words '*A* is true' is distinct from the proposition *A* itself." Bernard Bolzano, *Paradoxes of the Infinite* (London: Routledge and Kegan Paul, 1950), §13, pp. 84-85; cf. Rucker, 50.

113 Bolzano, *Paradoxes*, p. 84.

114 For the delineation of this principle as implicit in Cantor's view of sets I am indebted to Rudy Rucker, who calls it "the *genetic formation of sets principle*" (*Infinity and the Mind*, p. 208).

*Cantor*: That is true. I mentioned it in a letter to Dedekind a couple of years before Burali-Forti published his argument. It depends on the simple technical idea of a well-ordering: a multiplicity is called *well-ordered* if it fulfills the condition that every sub-multiplicity has a first element; such a multiplicity I call for short a *sequence*. Now I envisage the system *of all numbers* and denote it Ω. The system Ω in its natural ordering according to magnitude is a "sequence". Now let us adjoin 0 as an additional element to this sequence, and certainly if we set this 0 in the first position then Ω′ is still a sequence:

$$0, 1, 2, 3, \ldots \omega_0, \omega_0 + 1, \ldots, \gamma, \ldots,$$

of which one can readily convince oneself that *every* number occurring in it is the [*ordinal number*][115] of the *sequence of all its preceding elements*. Now Ω′ (and therefore also Ω) cannot be a consistent multiplicity. For if Ω′ were consistent, then as a well-ordered set, a number δ would belong to it which would be greater than all numbers of the system Ω; the number δ, however, also belongs to the system Ω, because it comprises *all* numbers. Thus δ would be greater than δ, which is a contradiction. Thus *the system Ω of all ordinal numbers is an inconsistent, absolutely infinite multiplicity*.[116]

*Leibniz*: So you rejected my proof that there is no number of all numbers, which I showed to follow from the axiom that the whole is greater than the part, since this number would be both equal to and greater than itself. Yet now you wish to persuade me that there is no ordinal number Ω of all ordinals, on the grounds that Ω, like Ω′, is a multiplicity necessarily greater than any of its parts, of which it is itself one—that is, that the ordinal number belonging to it would be both equal to and greater than itself!

*Cantor*: This shows that not every multiplicity of elements can be regarded as a set.

*Leibniz*: How then do you define a set?

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[115] Cantor had "*type* [*typus*]", defined as follows: "if the simply ordered multiplicity is a *set*, then I understand by *type* the general concept under which it stands, and also under which all and only *similar* ordered sets stand… If a sequence *F* has the character of a set, then I call the type of *F* its '*ordinal number*', or '*number*' for short." (Letter to Dedekind, 28th July, 1899, *GA* 444)
[116] *Ibid.*, *GA* 444-5.

*Cantor*: By a 'manifold' or 'set' I understand in general any 'many' [*Viele*] that can be thought of as one [*Eines*], that is, any domain of determinate elements which can be united into a whole through a law.[117]

*Leibniz*: This agrees well enough with my definition of a whole or *aggregate*: If, when several things are posited, by that very fact some unity is immediately understood to be posited, then the former are called *parts*, the latter a *whole*. Nor is it even necessary that they exist at the same time, or at the same place; it suffices that they be considered at the same time. Thus from all the Roman emperors together, we construct one aggregate.[118]

*Cantor*: But you countenance only finite aggregates?

*Leibniz*: No. The above definition is valid also for infinite aggregates, provided only they are united into a whole by the mind. Take, for instance, an infinite series. Even though the sum of this series cannot be expressed by one number, and yet the series is produced to infinity, nevertheless, since it consists of one law of progression, the whole is sufficiently perceived by the mind.[119] Note that this only involves the infinite in its distributive mode: given the first term and the law of the series, every subsequent term is determined. But this does not require that the mind be able to view all the terms as a collection.

*Cantor*: Your talk of a law of progression accords well with my own intentions in defining sets in terms of a law. For when I say a set is any domain that can be united into a whole through a law, I am thinking primarily in terms of the concept of well-ordering I just explained.[120] Thus I hold that in a similar way to how [the whole real

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[117] Cantor, "Grundlagen (1883)", *GA* 204. Cantor gives a similar definition in 1895: "By a 'set' we understand every collection into a whole *M* of determinate, well-differentiated objects *m* of our intuition or our thought (which are known as the 'elements' of *M*)." Cantor, "Beiträge",1895, *GA* 282. Cf. Hallett, p. 33.

[118] *LoC* 271 (A VI iv 627).

[119] Leibniz, GM.iv.120; translated from Yvon Belaval, p. 324.

[120] This is argued persuasively by Shaughan Lavine in his *Understanding the Infinite*, pp. 84-86. He observes that although the above passage has seemed to many like a classic statement of the Comprehension Principle, it continues: "I believe that in this I am defining something which is related to the *Platonic* ειδοσ or ιδεα, as well as to that which Plato in his dialogue *Philebus or the Highest Good* calls the μιχτον. He counterposes this to the απειρον, i.e. the unlimited, indeterminate, which I call the improper infinite, as well as to the περασ, i.e. the limited, and explains it as an ordered "mixture" of the latter two." Quoting relevant passages from the *Philebus* in support, Lavine argues that "Cantor's typical use of the word *law* in the *Grundlagen* is

number] ν is an expression for the fact that a certain finite number [*Anzahl*] of unities is unified into a whole, one may conceive ω to be the expression of the fact that the whole domain (I) [of the positive real whole numbers] is given in its natural succession according to law.[121]

*Leibniz*: I once held a similar view, as when I criticized Spinoza's claim that "there are many things which cannot be equated with any number". For if the multiplicity of things is so great that they exceed any number, that is, any number assignable by us, this multiplicity itself could be called a number, to wit, one that is greater than any assignable number whatever.[122] But shortly afterwards I came to the view I articulated above, that there is no number of all numbers at all, and that such a notion implies a contradiction.[123] That is, an infinity of things is not one whole, or at any rate not a true whole.[124] For when it is said that there is an infinity of terms, it is not being said that there is some specific number of them, but that there are more than any specific number.

*Cantor*: I agree that if one allows every well-defined infinite collection to count as a totality, then a contradiction can be obtained. That is why I do not allow such inconsistent multiplicities as the set of all the ordinals Ω or the totality of all cardinals to enter into set theory. If, on the other hand, the totality (*Gesamtheit*) of elements of a multiplicity can be thought without contradiction as "being together", so that their collection into "*one* thing" is possible, I call it a *consistent multiplicity* or a "set".[125]

*Leibniz*: I agree, one cannot simply assume the existence of a set or any other arbitrarily defined entity without first establishing its possibility. One needs what I call a real definition.

*Cantor*: What is that, may I ask?

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"natural succession according to law", which suggests quite a different picture: a 'law' is for Cantor a well-ordering or 'counting'… ".

[121] Cantor, "Grundlagen (1883)", *GA* 195.

[122] Leibniz, "Annotated Excerpts from Spinoza" (Feb-April 1676), *LCC* 111.

[123] Leibniz, *Pacidius Philalethi*, (1676), *LCC* 179.

[124] Leibniz, "Infinite Numbers," (April 1676), *LCC* 101. Cf. Leibniz, *New Essays*, p. 159: The idea of an infinite number is absurd, "not because we cannot have an idea of the infinite, but because an infinite cannot be a true whole."

[125] Cantor, Letter to Dedekind, 28 July 1899, *GA* 443.

*Leibniz*: A real definition is one according to which it is established that the defined thing is possible, and does not imply a contradiction. For if this is not established for a given thing, then no reasoning can safely be undertaken about it, since, if it involves a contradiction, the opposite can perhaps be concluded about the same thing with equal right. And this was the defect in Anselm's demonstration, revived by Descartes, that a most perfect or greatest being must exist, since it involves existence. For it is assumed without proof that a most perfect being does not imply a contradiction; and this gave me the occasion to recognize what the nature of a real definition is. So causal definitions, which contain the generation of the thing, are real definitions as well,[126] as are recursive definitions in mathematics. This is why the example I gave above of infinite division is apposite. The transfinite, however, is not susceptible to recursive definition.

*Cantor*: Consistency may be assumed if we know of no reason to deny it.

*Leibniz*: Then allow me to give another example. We are agreed, I believe, that monads are the true unities underlying all perceived phenomena, and that there are actually infinitely many of them.

*Cantor*: Yes. As I was saying earlier, I maintain that the *power* of *corporeal matter* is what I have called in my researches the *first* power,[127] i.e. that the number of corporeal monads is $\aleph_0$, the first transfinite cardinal.

*Leibniz*: Then you must hold that their collection into one thing or whole is possible.

*Cantor*: True: we must think of *corporeal matter* at each *moment of time* under the *representation* of a point-set *P* of the first power, and the aetherial matter in the same space under the *representation* of the next occurring point-set *Q* of the second power.[128] Each of the sets *P* and *Q* must be a consistent *totality*, otherwise the whole notion of a transfinite cardinal would fail.

*Leibniz*: Call it a totality you will. I say that the collection of all substantial unities or the accumulation of an infinite number of substances, is, properly speaking not a

126 Leibniz, *LoC* 305-07 (A VI iv, 1617).
127 Cantor, "Theorie der Punktmengen (1885)", *GA* 276.
128 *Ibidem*.

whole any more than infinite number itself.[129] And I will prove it to you. For we both agree that there is an actual infinity of true unities, and these can be aggregated, at least in thought. Now suppose the aggregate of these unities is itself a unity; let us call this unity U. Now according to the principle of the genetic formation of sets that you mentioned above, this unity is not included in U itself. Yet if it is the aggregate of *all* unities, it must be included in itself. Therefore U is not a true unity. No entity that is truly a unity is composed of a plurality of parts.[130]

*Cantor*: Here I believe a distinction must be made. Your argument shows that the collection of all corporeal monads cannot itself be a corporeal monad. But it does not prove that matter cannot be a consistent collection of a different order. The first transfinite ordinal, remember, is not a natural (or inductive) number, and the same applies to the first transfinite cardinal. What I assert and believe to have demonstrated in my various works and earlier researches is that after the finite there is a *transfinite*, (which one can also call *suprafinite*), that is, an unlimited ascending scale of determinate modes, which by their nature are not finite but infinite, but which, just like the finite, can be determined by definite, well-defined *numbers* distinguishable from one another.[131]

*Leibniz*: I, on the other hand, believe that all the paradoxes of the infinite have their source in the same error, namely, that of treating an infinite collection of elements as a whole. Allow me to explain, by giving a brief synopsis of our argument, drawing together more tightly our main points of contention. You assume that all the natural numbers can be regarded as a collection or a completed whole, or "one finished thing". Without this assumption it would not be possible for there to be a first number after all of them, as you define $\omega_0$, the first transfinite ordinal number. I have objected that the number of all numbers implies a contradiction, since the number of numbers in the part, or proper subset, would be equal to the number of numbers in the whole collection. You have evaded this objection by denying the applicability of the part-whole axiom to infinite collections—or, as I would rather explain it, by redefining a *proper part* of a transfinite set *S* as a subset *P* of *S* that cannot be put into element by element correspondence with *S*. Yet in your proof of

[129] Leibniz, *Theodicy* 249, GP VI 232.
[130] Leibniz, (A VI iv 627/LoC 271).
[131] Cantor, "Grundlagen" (1883), *GA* 176; cf. Hallett, p. 39.

the actual infinitude of the multiplicity of all ordinal numbers Ω, you acknowledge that if Ω were a set or consistent totality, then a set Ω′ could be formed, of which Ω would be a part by this definition. But the number corresponding to Ω′ would not only be greater than that corresponding to Ω, but also equal to it: the whole would be equal to the part, a contradiction. You therefore declare Ω to be an *absolutely infinite* or *inconsistent multiplicity* or one that it is impossible to construe as a unity, as "one finished thing". I would suggest that consistency demands either that every well-ordered multiplicity of ordinals is a consistent totality (which, as we have seen, leads to a contradiction) or that none of them is.

Again, your diagonal argument proves that if the infinite multiplicity of natural numbers forms a consistent totality with cardinality $\aleph_0$, then (assuming the power set axiom) there is necessarily a totality which is greater, to which there corresponds the greater cardinal number $2^{\aleph_0}$ (which, if I understand correctly, cannot be proved equal to $\aleph_1$, the first cardinal greater than $\aleph_0$.) But for each such totality there will be such a totality greater than it, including one greater than the totality of all thinkable totalities. But this is self-inconsistent. So, therefore, is the original totality, $\aleph_0$, by *modus tollens*.

*Cantor*: If I understand you correctly, you are asking how I know that the well ordered multiplicities or sequences to which I ascribe the cardinal numbers $\aleph_0, \aleph_1, \ldots \aleph_\omega, \ldots \aleph_\aleph, \ldots$ are really "sets" in the sense of the word I have explained, i.e. "consistent multiplicities". Is it not thinkable that *these* multiplicities are already "inconsistent", and that the contradiction arising from the assumption of a "being together of all their elements" has *simply not yet been made noticeable*? My answer to this is that the same question can just as well be raised about finite multiplicities, and that a careful consideration will lead one to the conclusion that even for finite multiplicities *no* "proof" of their consistency is to be had. In other words, the fact of the "consistency" of finite multiplicities is a simple, unprovable truth; it is (in the old sense of these words) "the axiom of finite arithmetic". And in just the same way, the "consistency" of those multiplicities to which I attribute the alefs as cardinal numbers is "the axiom of extended transfinite arithmetic."[132]

[132] Cantor, Letter to Dedekind, 28th August, 1899; *GA* 447-8; cf. Rucker, p. 254.

*Leibniz*: I am not so sure that it is impossible to demonstrate the consistency of finite multiplicities. But in any case, we know of no inconsistent finite multiplicities, whereas you admit there are inconsistent infinite ones.

*Cantor*: That may be so, but, notwithstanding your objections, I entertain no doubts about the truth of the transfinite, which with God's help I have recognized and studied in its diversity, multiformity and unity for so long.[133] My theory stands as firm as a rock; every arrow directed against it will quickly return to its archer.[134]

*Leibniz*: How can you be so sure?

*Cantor*: Because I have studied it from all sides for many years; because I have examined all objections which have ever been made against infinite numbers; and above all, because I have followed its roots, so to speak, to the first infallible cause of all created things.[135]

*Leibniz*: Suffice to say that God alone is infallible. I have enjoyed our conversation, which I am sure we will continue on another occasion. Good day, to you, most eminent Sir!

[133] Cantor , Letter to P. Ignatius Jeiler, Whitsun 1888; *Nachlass* VI, 169; quoted from Hallett, p. 11.
[134] Cantor, quoted from Dauben, p. 298.
[135] *Ibidem.*

**Afterword**

Since Leibniz was too gracious to press his point further against Cantor, perhaps I may summarize the argument on his behalf, and say a few words on what I take to be its significance. If the above is a fair statement of his position, then I think it must be granted that

1) Leibniz's construal of the infinite as an actual infinite, syncategorematically understood, and as a distributive whole but not a collection or set, constitutes a perfectly clear and consistent third alternative in the foundations of mathematics to the usual dichotomy between the potential infinite (intuitionism, constructivism) and the transfinite (Cantor, set theory).

   [This should be contrasted with the following passage from one of the foremost modern proponents of intuitionism, Michael Dummett: "In intuitionist mathematics, all infinity is potential infinity: there is no completed infinite. … [This] means, simply, that to grasp an infinite structure is to grasp the process that generates it, that to refer to that structure is to refer to that process, and that to recognise the structure as being infinite is to recognise that the process will not terminate."[136] To this it may be replied on Leibniz's behalf: there is also an actual infinite which is not a completed whole. All the terms of an infinite series are actually given once the law and the first term are given, provided this "all" is not understood collectively.]

2) This conception of the infinite as not involving infinite number is appropriate to Leibniz's conception of the actual infinite division of matter; whereas Cantor's transfinite is not, since one cannot get to an $\omega^{th}$ part by recursively dividing.

   [This can be compared with some criticisms of Leibniz's philosophy of the infinite in a recent article by one of his ablest and most intelligent commentators, Samuel Levey.[137] Levey charges that "the deep cause of the difficulty in Leibniz's theory of matter—what leads him into error and what prevents him from seeing it—is his

[136] Michael Dummett, *Elements of Intuitionism* (Clarendon Press : Oxford, 1977), 55-56. Cf. Lavine, *op cit.*: "The intuitionist endorses the actual finite, but only the potentially infinite… The constructivist of my first example endorses the actual denumerable, but only the potentially non-denumerable." But Hidé Ishiguro (private communication) dissents from Dummett's view, holding that Brouwer's conception of the infinite is more nearly syncategorematic.
[137] Samuel Levey, "Leibniz's Constructivism", *op. cit*. fn. 41, esp. pp. 152ff.

constructivism." (156) In a nutshell, Levey sees Leibniz as endorsing an *actualism* in his account of matter that is undermined by his tendency toward a *constructivist* view of the infinity of numbers. "An infinity of parts comes to view only from the outside perspective, where the completed infinity can be seen all at once... [A]s it falls outside the series of finite levels but encompasses them all, we might call it the 'omega level'... (156) To the constructivist about the infinite, however, *there is no omega level.* Infinity is always potential and incomplete... Leibniz lets his constructivism come between him and the parts of matter. By thinking in constructivist terms about the division of matter into parts, he loses sight of the omega level and the problems that should appear most vividly there." (157) But the above argument shows that the actual, distributive infinite is understood from the inside: there is no omega level of the division of matter, but this does not make it any less actual. Rather than "dividing into two isolated accounts, the actual infinity of nature and the constructive infinity of mathematics" (157), Leibniz gives an integrated account that avoids equally the paradoxes of the infinite and the shortcomings of constructivism.[138]

3) Through this example of the actually infinite division of matter one can see that recursive connection is closely linked to Leibniz's demand for real definitions, according to which the possibility of the thing defined must be established in the definition.[139] Thus in Leibniz's causal definition of an infinite aggregate, each element is generable by recursion from some initial element (body).

[This aligns Leibniz's philosophy of mathematics closely with constructivism and intuitionism, as Levey has observed. But the same example of infinite division shows that the generation *need not be thought of as a process in time*. So long as there is a law according to which further elements are generated (as in the case of the Euclidean generation of primes), an infinity of elements can be

[138] Here I would not be understood as rejecting all of Levey's analysis. I believe he is right in seeing a tendency in Leibniz toward a constructivist view of numbers, but do not agree that this results in any vacillation about the actuality of infinity. Also, much of what Levey has to say about limits, the division of the continuum, and the origins of Leibniz's monadism is pure gold. But that is another matter, and there is not the space to go into it here.
[139] This was already noted long ago by Nicholas Rescher, in his *The Philosophy of Leibniz* (Englewood Cliffs: Prentice Hall, 1967), p. 105: "Leibniz, on wholly logical grounds, rejects the notion of infinite number, holding that a definition must involve a proof of the possibility of the thing defined, as he had maintained earlier against Descartes' ontological proof."

understood without any commitment to either an infinite collection (Weierstraß, Cantor) or an unending temporal process (Aristotle, Kant, Brouwer, Weyl).]

It was further argued that:

4) Cantor's diagonal argument does not constitute a conclusive proof that there are more reals than natural numbers, since it depends on the premise that the real numbers form a totality, a collection such that none are left out; and this is established by the Power Set Axiom only on the hypothesis that the natural numbers form such a totality. Similarly, 1-1 correspondence between the elements of two multiplicities does not establish that they constitute equal sets without the assumption that those infinite multiplicities are indeed consistent totalities.

5) Leibniz's claim that the universe, or any other collection of all unities, cannot itself be a unity, can be proved in an entirely analogous way to Cantor's proof that there is no ordinal number of all ordinal numbers; and that this result, together with the inapplicability of the transfinite to Leibnizian aggregates (and, one might add, the unprovability of the continuum hypothesis), precludes the kind of application of the transfinite to physics that Cantor had envisaged.

6) Leibniz's insistence on the part-whole axiom, rejected by Bolzano and Cantor for infinite sets, can be seen to be justified even in Cantorian set theory, provided *whole* is identified with Cantor's *consistent totality* or *set*, and *part* is re-identified as a subset that cannot be put into 1-1 correspondence with the whole. Although this allows one to avoid Galileo's paradox, since the set of even numbers is not a "part" of the set of natural numbers in this sense, it cannot avoid the paradox of the number of all ordinals without some independent criterion for disqualifying the multiplicity of all ordinals from forming a set.

Whether this Leibnizian approach to the foundations of mathematics is adequate to the construction of any interesting mathematics is a topic for another project. Obviously, it will be crucial to see what kind of account of continuum can be sustained in opposition to the point-set foundation, since the Leibnizian approach eschews infinite sets. [140]

[140] I am indebted to several colleagues for their encouragement and generous critical feedback on earlier drafts of this dialogue, most especially to Wayne Myrvold, Massimo Mugnai, Dean Buckner, Bill Harper and Jim Brown.