Critical Studies/Book Reviews

David Hilbert. ***David Hilbert's lectures on the foundations of geometry, 1891–1902.*** Michael Hallett and Ulrich Majer, eds. David Hilbert's Foundational Lectures; 1. Berlin: Springer-Verlag, 2004. ISBN 3-540-64373-7. Pp. xxviii + 661.[†]

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1. Introduction

This volume, the result of monumental editorial work, contains the German text of various lectures on the foundations of geometry, as well as the first edition of Hilbert's *Grundlagen der Geometrie*, a work we shall refer to as the *GdG*. The material surrounding the lectures was selected from the Hilbert papers stored in two Göttingen libraries, based on criteria of relevance to the genesis and the trajectory of Hilbert's ideas regarding the foundations of geometry, the rationale, and the meaning of the foundational enterprise. By its very nature, this material contains additions, crossed-out words or paragraphs, comments in the margin, and all of it is painstakingly documented in notes and appendices. Each piece is preceded by informative introductions in English that highlight the main features worthy of the reader's attention, and place the lectures in historical context, sometimes providing references to subsequent developments. It is thus preferable (but by no means necessary, as the introductions alone, by singling out the main points of interest, amply reward the reader of English only) that the interested reader be proficient in both German and English (French would be helpful as well, as there are a few French quotations), as there are no translations.

Chapter 1 consists of manuscript notes in Hilbert's own hand for a course on projective geometry held in Königsberg in the summer semester of 1891; Chapter 2 is his own notes for a course on the foundations of geometry in Königsberg, planned for the summer of 1893, but held during the summer semester of the following year; Chapter 3 contains Hilbert's notes for two *Ferienkurse* for high-school teachers which took place in Göttingen in 1896 and 1898. Perhaps the most interesting of all, is Chapter 4. It consists of two versions of notes for a course on the

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foundations of Euclidean geometry, delivered during the winter semester 1898/1899 in Göttingen. These notes form the basis of the *GdG* of 1899, and contain a wealth of information of a strictly mathematical nature, which never made it into any edition of the *GdG*, and a wealth of statements relevant to both Hilbert's motivations for proceeding the way he does and his philosophical views on the nature of the axiomatic enterprise. Unlike the Hilbert of *GdG*, with his extreme concision, the Hilbert we encounter in these lectures is talkative and opens a window to his mathematical mind's house, allowing us glimpses of his aims and thoughts, and their gradual change with the passage of time. Chapter 5 contains the text of the first edition of the *GdG*, — a book that has become, like many first editions, a rarity[1] — referred to as the *Festschrift*, as it was written on the occasion of the unveiling of the Gauss-Weber monument. The introduction to Chapter 5 refers succinctly to the various editions that followed, without going into details regarding the changes. The reader who would like to have a synopsis of all these changes, all the way up to the eighth edition, should consult [Hilbert, 1971]. Chapter 6, the last one, contains a version of Hilbert's lectures on the foundations of geometry delivered during the summer semester of 1902. This offers all interested researchers material that has been first interpreted in [Toepell, 1986], but had been available only through selected quotations.

We will focus in this review on the following topics: (i) Hilbert's view on the process of axiomatization and on the nature of geometry, as can be read from the material in the lectures; (ii) the specific questions raised in the lectures as providing a road map for the advances in the foundations of geometry obtained during the twentieth century and beyond; (iii) interesting material that can be found in the lectures, but not in *GdG*; and (iv) the legacy of the *GdG*.

2. The Process of Axiomatizaton and the Nature of Geometry

Ad (i). First of all, one finds that Hilbert is very far in practice from formalism in the *wide* sense (as emphasized by Corry [2000; 2002; 2006] repeatedly) in which mathematics becomes a game with strict rules that manipulates meaningless formulas. His formalism is only a tool by which he tried to prove the consistency of arithmetic. In the practice of the axiomatization of geometry he did what his predecessors, Pasch, Peano, Pieri, had done as well, namely imagine an axiom system that does not require an understanding of either the entities or the predicates of the language in which it is expressed, that would go beyond having grasped their

[1] It will likely become even more of a rarity, as university libraries in the United States have been discarding it, given that it was superseded by *newer* editions. One such discarded copy of the *Festschrift* was saved in the twenty-first century by Alexander Soifer for US$1.

characteristics as stipulated by the axioms of the axiom system. In fact, one is surprised to find that, although his empiricism is not as strict as that of Pasch — whose empiricism has recently been the subject of intense scrutiny [Gandon, 2005; Schlimm, 2010] — he shares with Pasch the view that geometry corresponds to aspects of our experience. On page 171 he notes, for example, that the Parallel Postulate and the Archimedean Axiom do not have the *empirical*[2] character of the other axioms that allow construction by a finite number of experiments. On page 302 we read that 'geometry is the the most accomplished natural science',[3] on page 303 that the task of the foundations of geometry is that of 'a logical analysis of our intuition's capabilities',[4] on page 391 that 'the geometric theorems are never true in nature in full exactness, since the axioms are never fulfilled by objects', but that this should not be taken to be a flaw, for it is a characteristic of theories, for 'a theory, that would coincide in all details with reality, would be no more than an exact description of the objects'. Another remarkable feature of Hilbert's approach to the axiomatization of Euclidean geometry is the high regard which he has for Euclid, whom he is proud to correct or outwit sometimes. The entire enterprise, of turning geometry back into the deductive art it once was in ancient Greece, away from the *ars calculandi* that the analytic geometry 'premisses' of another age had turned geometry into,[5] appears to have taken a page from Johann Joachim Winckelmann, whose credo was 'Der einzige Weg für uns, groß, ja, wenn es möglich ist, unnachahmlich zu werden, ist die Nachahmung der

[2] Hilbert's own emphasis.

[3] 'Geometrie ist die vollkommenste Naturwissenschaft.'

[4] 'eine logische Analyse unseres Anschauungsvermögens'. On the first page of the *GdG* itself, we find that the task of the book amounts to 'the logical analysis of our spatial intuition.' ('die logische Analyse unserer räumlichen Anschauung' (p. 436))

[5] 'Mit diesen Prämissen ist dann sofort aus der *Geometrie eine Rechenkunst* geworden' (p. 222). By creatively restoring elementary geometry to its former state, to an undertaking confined to its own language, without algebraic admixtures, Hilbert finds a solution to the critique expressed often by Newton of the Cartesian approach (for an in-depth look at Newton's pronouncements regarding geometry and algebra see [Guicciardini, 2009]), such as in: 'Anyone who examines the constructions of problems by the straight line and circle devised by the first geometers will readily perceive that geometry was contrived as a means of escaping the tediousness of calculation by the ready drawing of lines. Consequently these two sciences ought not to be confused. The Ancients so assiduously distinguished them one from the other that they never introduced arithmetical terms into geometry; while recent people, by confusing both, have lost the simplicity in which all elegance of geometry consists' [Newton, 1673/83, p. 428]. Hilbert expresses a similar view in the 1898–99 lectures: 'However, if science is not to fall prey to a *sterile formalism*, then it *must reflect* in a later stage of its development *on itself*, and should at least analyse the foundations that led to the introduction of number.' ('Aber, wenn die Wissenschaft nicht einem *unfruchtbaren Formalismus* anheimfallen soll, so wird sie auf einem späteren Stadium der Entwicklung sich wieder auf sich *selbst besinnen müssen* und mindestens die Grundlagen prüfen, auf denen sie zur Einführung der Zahl gekommen ist.' (p. 222))

Alten.'[6] Referring to the relationship between projective and Euclidean geometry, Hilbert finds that 'in projective geometry one takes recourse from the very beginning to intuition', whereas in the axiomatization of Euclidean geometry it is precisely 'intuition that we want to analyze, to build it afterwards anew out of its constituent parts'.[7] Perhaps surprising in view of Hilbert's later positions, but understandable in a time in which the notion of first-order logic was still more than two decades away, is the fact that, in 1898–1899, the logical background against which an axiom system is set up, consists not just of logic, but rather of 'the laws of pure logic as well as the entire arithmetic',[8] with a reference to Dedekind's *Was sind und was sollen Zahlen?* regarding the relationship between logic and arithmetic, as if in agreement with the logicist stance on the issue.[9] Something resembling arithmetic, however, is used only in the formulation of the Archimedean Axiom, for one of the aims of the *GdG* is to introduce number into geometry in a purely geometrical manner, for 'in any exact science, a most highly prized aim is the introduction of numbers'.[10] Hilbert's obsession with the 'introduction of number' ('Einführung der Zahl'), which he finds (on p. 282) to be one of those 'true, genuine wonders' Lessing's Nathan the Wise referred to as miraculously becoming routine, leads us to our next topic, (ii).

3. A Road Map for the Advances in the Foudations of Geometry

We will look at the main questions Hilbert asked in his lecture notes, and the way in which they dominated much of the research in the axiomatic foundation of geometry. Since these questions revolve around two configuration theorems, Desargues and Pappus, we first state them. The projective form of the Desargues theorem (to be denoted by *Des*) states that 'If ABC and $A'B'C'$ are two triangles, such that the lines AA', BB', CC' meet in a point O, then the intersection points X, Y, Z of the line pairs AB and $A'B'$, BC and $B'C'$, CA and $C'A''$ are collinear.' (The affine form (to be denoted by ${}^{a}Des$) states that if two of the line pairs above are parallel, then so is the third pair.) The projective form of Pappus

[6] 'The only way in which we could become great, and even, if at all possible, inimitable, is the imitation of the ancients.'

[7] 'in der projectiven Geometrie appelliert man von vornherein an die Anschauung, wogegen wir ja die Anschauung analysieren wollen, um sie dann sozusagen aus ihren Bestandteilen wieder aufzubauen.' (p. 303)

[8] 'Als gegeben betrachten wir die Gesetze der reinen Logik und speciell die ganze Arithmetik.' (p. 303)

[9] Ferreirós [2009] believes that Hilbert actually was a logicist at the time. For a comparison of Dedekind's and Hilbert's early approach to foundational matters see [Klev, 2011].

[10] 'Nun ist in der That in jeder exakten Wissenschaft die Einführung der Zahl ein vornehmstes Ziel.' (p. 222)

(to be denoted by *Papp*) states that 'If A, B, C and A', B', C' are two sets of three collinear points, lying on different lines, then the intersections points X, Y, Z of the line pairs AB' and $A'B$, AC' and $A'C$, BC' and $B'C$ are collinear'. (The affine version (to be denoted by $^a Papp$) states that if two of the line pairs above are parallel, then so is the third pair.) Notice that the projective forms are universal statements in terms of points, lines, and incidence alone, as the existence of the points of intersection of the respective lines is part of the hypothesis of these configuration theorems. Hilbert's recurring questions are:

(1) Is $^a Des$ provable with the help of the congruence axioms alone? (p. 172)
(2) Does $^a Papp$ (referred to as Pascal's theorem throughout) follow from the congruence axioms together with $^a Des$? (p. 174)
(3) Does $^a Des$ follow from the congruence axioms and $^a Papp$? (pp. 174, 175)
(4) Prove $^a Papp$ (and $^a Des$) in the plane based only on axioms in the groups I, II, and III (*i.e.*, on the basis of the axioms for absolute geometry, without using the Parallel Postulate) (pp. 284, 392); in another form: $^a Papp$ 'arises by the elimination of the congruence axioms, indeed [$^a Papp$] is the sufficient condition that ensures that a definition of congruence is possible'. (p. 261)[11]
(5) $^a Des$ must hold if a plane is to be part of space.[12] Is it also a sufficient condition for this to happen? (p. 318)
(6) 'The Desargues theorem is the result of the elimination of the spatial axioms from I and II' (p. 318).[13]

Also in connection with $^a Des$, Hilbert expressed a much broader[14] concern that, we are told on the last page of the *GdG*, is the 'subjective form' of the main concern of the axiomatic foundation of geometry,[15] namely

[11] Here Hilbert refers to both (4) and (6) in the same sentence. It reads: 'So wie der Desargues gewissermassen die Elimination der räumlichen Axiome ist, so entsteht der Pascal durch die Elimination der Congruenzaxiome und zwar ist der Pascal auch die hinreichende Bedingung dafür, dass eine Congruenzdefinition möglich ist.' (p. 261)

[12] Notice that Hilbert had proved the necessity (and sufficiency) of $^a Des$ for a plane to be part of three-dimensional space only in the affine case. That any plane in an ordered space (*i.e.*, in a model of Hilbert's incidence and order axioms) has to satisfy Desargues was first shown in [Pasch, 1882, pp.46–55] (see also, [Abellanas, 1946])

[13] Since there are no proper spatial axioms in the group II of order axioms, Hilbert must have meant here just group I of incidence axioms.

[14] See also [Arana, 2008; Arana and Detlefsen, 2011, pp. 6–7].

[15] 'Der Grundsatz, demzufolge man überall die Principien der Möglichkeit der Beweise erläutern soll, hängt auch aufs Engste mit der Forderung der "Reinheit" der Beweismethoden zusammen, die von mehreren Mathematikern der neueren Zeit mit Nachdruck erhoben worden ist. Diese Forderung ist im Grunde nichts Anderes als eine subjektive Fassung

(7) the concern with the purity of the method (‘Reinheit der Methode’), stemming from the fact that ^{a}Des, although a statement belonging to plane geometry, is usually proved by recourse to the geometry of space, in which the plane is supposed to be embedded.

Here Hilbert criticizes the means of proof (although he shows that ^{a}Des cannot be proved from the plane incidence axioms — not even if the congruence axioms, with the ‘Side-Angle-Side’ congruence axiom omitted, and the Euclidean Parallel Postulate are allowed in the proof) and states that ‘In modern mathematics one exercises this sort of critique very often, whereby we notice the tendency to preserve *the purity of the method*, *i.e.*, to use in the proof of a theorem, if possible, only the auxiliary means that the content of the theorem imposes upon us.’[16]

(8) Is the Archimedean axiom needed to prove the Legendre theorems (regarding two facts, namely that if the sum of the angles of one triangle is π, then it is π for all triangles, and that the sum of the angles of every triangle does not exceed π)? (p. 392)

All of these questions were prompted by the interest in ‘introducing number’ into geometry, which Hilbert finds not only of interest for purity of the method’s sake, but also as a means of transferring the consistency problem of Euclidean geometry to that of a ‘number system’, a certain field — or a certain class of fields — in today’s language,[17] for the Desargues theorem (in both forms) is a configuration theorem (*i.e.*, a universal statement regarding points, lines, and incidence) that ensures that the plane can be coordinatized by means of skew fields (being essential in proving the associativity of multiplication), and the Pappus theorem (in both forms) is a configuration theorem ensuring the commutativity of multiplication.

These problems have motivated a great part of the axiomatic foundations in the twentieth century and beyond, and have produced some of

des hier befolgten Grundsatzes. In der That sucht die bevorstehende Untersuchung allgemein darüber Aufschluss zu geben, welche Axiome, Voraussetzungen oder Hilfsmittel zum Beweise einer elementar-geometrischen Wahrheit nötig sind’ (p. 523; last page (page 90) of the *GdG*).

[16] In der modernen Mathematik wird solche Kritik sehr häufig geübt, wobei das Bestreben ist, *die Reinheit der Methode* zu wahren, d.h. beim Beweise eines Satzes wo möglich nur solche Hülfsmittel zu benutzen, die durch den Inhalt des Satzes nahe gelegt sind. (pp. 315–316)

[17] The problem regarding the consistency of solid Euclidean geometry is addressed both in the 1898–99 lectures: ‘Diese [die analytische Geometrie] ist möglich, weil die Eigenschaften der reellen Zahlen sich nicht einander widersprechen, sondern alle miteinander verträglich sind.’ (p. 282), and in *GdG* — in Kap. II, §9 (pp. 454–455) — where the consistency of Euclidean geometry is shown by pointing to the absence of contradiction inside the Pythagorean closure of the field of rational numbers, or put differently, by assuming that the theory of Archimedean ordered Pythagorean fields is consistent.

the deepest and most astonishing results pure geometry has ever seen.[18] First, let us look at the legacy of questions (1)–(4). That *Papp* implies *Des* in projective planes (*i.e.*, if one assumes that every line is incident with at least three points, and that any two lines intersect, in addition to Hilbert's plane incidence axioms) was proved by Hessenberg in [1905a] (a gap in the proof was closed in [Cronheim, 1953]). Whether an ordered plane satisfying *Papp* must satisfy *Des* as well (*i.e.*, whether the plane incidence and order axioms together with *Papp* imply *Des*) is still not known. The first contribution regarding the effect of the congruence axioms (in the absence of the Parallel Postulate or the hyperbolic version thereof) on *Papp* and *Des*, is in the ground-breaking paper by Hjelmslev [1907]. The central realization of that work was that line-reflections have certain properties that are independent of any assumption regarding parallels and thus absolute. Line-reflections, and in particular the central *three-reflection theorem*, stating that the composition of three reflections in lines that have a common perpendicular or a common point must be a line-reflection, had appeared earlier in [Wiener, 1893; Schur, 1899; Hilbert, 1903a; Hessenberg, 1905b], but in these works line-reflections were not treated independently of the particular geometry in which they were defined (Euclidean, hyperbolic, or elliptic), as they were by Hjelmslev, who carried on this research in [Hjelmslev, 1929]. Many more — whose contributions are chronicled in [Karzel and Kroll, 1988] — have helped understanding geometry in terms of line-reflections as primitive notions. They helped remove assumptions regarding both the order of the plane and the free mobility of the plane (*i.e.*, the possibility of transporting segments on any given line). What is left after the removal work was done consists of the three-reflections theorem, together with very basic axioms stating that there are at least two points, that there is exactly one line incident with two distinct points, that perpendicular lines intersect, and that through every point there is a perpendicular to any line, which is unique if the point and the line are incident. The final touch in carving this austere axiom system came from Friedrich Bachmann [1951], who showed that two axioms proposed by Hilbert's student Arnold Schmidt [1943] are superfluous. The axiom system can also be conceived as one expressed in terms of orthogonality alone (see [Pambuccian, 2007a]), and, in the non-elliptic case, in terms of incidence and segment congruence (or just in terms of the latter alone), as required by the original statement of (4), as shown in [Sörensen, 1984].[19] The group-theoretical and the traditional geometric axiomatizations are logically equivalent,

[18] These problems are still open for many Cayley-Klein geometries. (see [Struve and Struve, 1985; 2004; 2010].)

[19] Something that Hilbert apparently considered impossible in 1898–99: 'Der umgekehrte Weg, die Kongruenzaxiome und -sätze mit Hülfe des Bewegungsbegriffs zu

as spelled out in [Pambuccian, 2005b]. What is remarkable about this austere axiom system for structures called *metric planes* is that it is strong enough to prove both *Papp* and *Des*, metric planes being embeddable in projective planes coordinatized by fields of characteristic $\neq 2$ and endowed with an orthogonality relation extending that of the metric plane [Bachmann, 1973]. In fact, an even weaker axiom system, in which the three-reflections theorem is weakened, implies both *Des* and *Papp*. That it implies *Des* was shown by Sperner [1954], who set out to answer precisely the question asked by Hilbert in (1), going beyond the original question by asking for minimal 'congruence' assumptions, expressed in terms of line-reflections.[20] It turns out that even Sperner's structures, which were thoroughly studied in [Lingenberg, 1959; 1960/1961; 1965], satisfy *Papp* (see [Karzel and Kroll, 1988, pp. 181–182] and [Lingenberg, 1979]), and they can be embedded in projective planes satisfying *Papp*. A far-reaching generalization of Sperner's structures was proposed in [Schröder, 1984], and studied in depth in [Saad, 1988]. No axiom system for some absolute geometry is known that would satisfy only *Des*, but not *Papp*. There are several axiom systems [Schütte, 1955a; 1955b; Naumann, 1956; Naumann and Reidemeister, 1957; Quaisser, 1975; Szmielew, 1983; Kusak, 1987] that provide in the Euclidean case a precise answer to (1) and (2), *i.e.*, provide orthogonality or congruence axioms that are strong enough to imply ^{a}Des but not $^{a}Papp$. In the same Desargues and Pappus area of concern, we find the following statements of Hilbert that 'Every configuration theorem can be proved by means of Pappus and Desargues using only the incidence axioms.'[21] This statement was included in the *GdG* and preserved in all future editions. It states that every configuration theorem true in ordered affine planes in which ^{a}Des and $^{a}Papp$ hold (later the requirement that ^{a}Des holds is dropped, with a reference to [Hessenberg, 1905b]) 'will always turn out to be a combination of the Desargues and Pappus theorems' (*GdG*, Kap. VI, §35, p. 511), and in later editions 'a combination of finitely many Pappus configurations'. If by 'configuration theorem' we are to understand a universal sentence in terms of points, lines, and point-line incidence, then it is not true that every configuration theorem true in ordered Pappian affine planes can be

beweisen ist falsch, da sich die Bewegung ohne den Kongruenzbegriff gar nicht definieren lässt.' (p. 335)

[20] In fact, given that its models can be embedded in projective planes over commutative fields without the requirement that the characteristic be $\neq 2$, the axiom system in [Sperner, 1954] answers a minimalist question very close to (1)–(4), namely that for 'congruence' axioms that would, together with the trivial incidence axioms, be just strong enough to prove *Papp* and every projective configuration theorem that can be proved from *Papp*, but no other configuration theorem.

[21] 'Aus Pascal und Desargues kann allein durch die Axiome der Verknüpfung jeder Schnittpunktsatz [...] bewiesen werden.' (p. 178)

derived from the incidence axioms for affine planes and ${}^{a}Papp$, for the obvious reason that Fano's axiom ('The three diagonal points of a complete quadrilateral are never collinear') must hold in ordered affine planes, and Fano's axiom is not true in all Pappian affine planes. This was certainly clear to Hilbert. Whatever he might have had in mind, his having emphasized the universal incidence theory of a theory expressed in a language with incidence and order, raised a general question with far-reaching consequences. Given a theory $\mathcal{T}_{BI}$ in terms of betweenness B and incidence I, find its universal incidence theory $(\mathcal{T}_I)_\forall$. With $\mathcal{T}_{BI}$ being the theory of ordered affine planes satisfying ${}^{a}Papp$, the question was answered in [Artin and Schreier, 1926]. The answer is: one needs to add to the trivial incidence axioms and to ${}^{a}Papp$ an axiom schema stating, in $(\mathcal{T}_I)_\forall$, that the sum of non-zero squares is never equal to zero. In the case of ordered affine planes satisfying ${}^{a}Des$ the answer was provided independently in [Pickert, 1951] and [Szele, 1952], and for the plain theory of ordered affine planes in [Kalhoff, 1988]. The results of Sperner [1938] and Joussen [1966] determine the theories $(\mathcal{T}_I)_\forall$ even if $\mathcal{T}_{BI}$ is the theory of ordered planes satisfying *Des* or just that of plain ordered planes (axiomatized by Hilbert's plane incidence and order axioms).

As far as (5) and (6) are concerned, Hilbert provides with Theorem 35 of *GdG* (Kap. V, §30, p. 505) a rather modest answer to the questions. For although he shows that Desarguesian ordered affine planes can be embedded in ordered affine spaces, and thus that ${}^{a}Des$ is indeed a sufficient condition for a plane to be a part of space, he does this in an algebraic manner. After coordinatizing the affine plane by a skew field, he adds a third coordinate to each point (x, y) of the plane, which becomes $(x, y, 0)$ in the affine space in which the plane is embedded. A proof *more geometrico* of Theorem 35 was provided in [Schor, 1904]. Purely geometric embeddings of any Desarguesian projective plane into a 3-dimensional projective space (without any order relation) were provided in [Levi, 1939; Fritsch, 1974; Herzer, 1975; Baldwin, 2013], the proof by Howard and Baldwin in [Baldwin, 2013, §4] being entirely in the spirit of classical projective geometry. The first step toward the solution of (5) in the case of ordered planes, was taken by Owens [1910], who showed that an ordered plane, in which a strong form of Desargues axiom (the point O does not need to be a proper point) and of its converse hold, can be embedded in a projective ordered plane satisfying *Des*. Unaware of Owen's work, Ruth Moufang [1931] showed that an ordered plane that satisfies the projective minor Desargues axiom (whenever all points involved belong to the ordered plane) can be embedded in a projective ordered plane, which in turn satisfies the projective minor Desargues axiom. Sperner [1938] completely solved Hilbert's problems (5) and (6) by showing that Desarguesian ordered planes can be embedded in projective Desarguesian ordered planes (it follows from results proved later, in [Skornyakov, 1949] and [Bruck and Kleinfeld, 1951], that Moufang's 1931 result implies Sperner's 1938 result), and thus that they are indeed part of ordered three-dimensional spaces as defined in *GdG*.

Problem (7) has opened up a large field of investigations, by asking for pure proofs for a specific result, which amounts to proofs proceeding from minimalist assumptions, both in the sense of the language employed and in the sense of a minimalist content of the axioms. This does not boil down to searching for a proof in absolute geometry for a theorem known to be true in Euclidean geometry, but rather aims at finding the right assumptions that are needed to prove a theorem. In Hilbert's own words

> By the axiomatic analysis of a mathematical truth I understand an investigation, which does not aim to discover new or more general theorems relative to that truth, but rather aims to clarify the position of that theorem inside the system of known truths and their mutual logical connections in such a way that one can indicate exactly which conditions are necessary and sufficient for justifying that truth.[22]

Perhaps the first such in-depth investigation was Hilbert's own [1903b] (to become Appendix II to *GdG* in later editions), followed by the search for the minimalist 'congruence' (in fact, orthogonality) axioms required to prove Pappus, resulting in the flurry of papers mentioned earlier regarding (4), by the in-depth analysis of the axioms required to prove the Möbius-Pompeiu theorem from [Barbilian, 1936] (see also [Pambuccian, 2009a]), by Coxeter's considerations on the purity of the proof of the Sylvester-Gallai theorem (see also [Pambuccian, 2009b]), by the marvelous result from [Bachmann, 1967] regarding the minimal assumptions required to prove the generalized concurrency of the altitudes of a triangle, by the analysis of the axioms needed to prove Euclid's Proposition I.1 in [Pambuccian, 1998a] (see also [Hartshorne, 2000, p. 373]), as well as in [Schröder, 1985; Fritsch, 1995; Pambuccian, 2003; 2005a; 2006; 2007b; 2010; Hociotă and Pambuccian, 2011; Pambuccian and Struve, 2009; Pambuccian, Struve, and Struve, forthcoming; Pambuccian, 2012], in which minimal assumptions in both primitive notions and axioms were found for specific geometric statements.[23] A concerted effort to find minimal assumptions, motivated by the 'purity of the method' concern,

[22] 'Unter der axiomatischen Erforschung einer mathematischen Wahrheit verstehe ich eine Untersuchung, welche nicht dahin zielt, im Zusammenhange mit jener Wahrheit neue oder allgemeinere Sätze zu entdecken, sondern die vielmehr die Stellung jenes Satzes innerhalb des Systems der bekannten Wahrheiten und ihren logischen Zusammenhang in der Weise klarzulegen sucht, daß sich sicher angeben lät, welche Voraussetzungen zur Begründung jener Wahrheit notwendig und hinreichend sind.' [Hilbert, 1903b]

[23] The assumptions in [Bachmann, 1967; Pambuccian, 2003; 2006; 2009b; 2010; Hociotă and Pambuccian, 2011, Pambuccian, 2012] are too weak to allow for an algebraization, providing one more reason why geometry cannot be reduced to algebra.

can be found in the papers [Andréka, Madarász, and Németi, 2006; Andréka, Madarász, Németi, and Székely, 2008; Madarász, Németi, and Székely 2006; Székely, 2010], focusing on very austere axioms needed to prove certain theorems in relativity.

The original concern in (7), regarding Desargues's plane nature and solid proof, is still [Arana and Mancosu, 2012; Baldwin, 2013] motivating reflections regarding various ways in which the purity of the method can be understood, and the relationship between plane and solid geometry.

As for question (8), it was answered by Max Dehn [1900] a short time after it was raised, yet the crowning achievement, which would allow a wide class of like-minded questions to be answered with a modest amount of algebraic skill, came with the algebraic characterization of all models of Hilbert's plane axioms of incidence, order, and congruence by Wolfgang Pejas [1961], a characterization made possible by the most sophisticated answers given to (2), in particular by Bachmann [1973].

4. Previously Unpublished Material in Hilbert's Lectures

Regarding (iii), it is surprising how much more is to be found in the lectures than in *GdG* or in any other published material. Hilbert provides a wealth of independence models for all sorts of axioms, starting with the incidence axioms (pp. 306–307) and each order axiom, including an independence model for the Pasch axiom (pp. 232, 311–312), which is shown to be independent of all the incidence axioms and all the linear-order axioms. The most intriguing among the questions on independence is the question regarding the independence of the symmetry axiom for the metric, *i.e.*, of the statement that segment AB is congruent to segment BA. Its intrinsic difficulty is responsible for the greatest confusion that can be found in the entire book. An independence model from all other axioms, excluding the Side-Angle-Side triangle-congruence axiom, is provided on page 286 (and the independence is mentioned again on p. 399), although Hilbert tries unsuccessfully to prove it twice, once on p. 109[24] and then again on pp. 320–321, where he gives up the attempt (after claiming that it can be proved by 'the introduction of a minor change'[25]), to re-introduce $AB \equiv BA$ as an axiom on p. 322 (it had been taken as an axiom on p. 319). Pages 218–219 of the editors' notes are devoted to this axiom, and on p. 218 the editors state that $AB \equiv BA$ 'is a simple consequence of the Triangle Congruence Axiom' (by which the Side-Angle-Side congruence axiom is meant). It is not clear what is meant by this comment, as no proof of $AB \equiv BA$ from a traditional axiom system is known. As far as

[24] We find the following mesmerizing insertion between the proposition and its proof: 'Dieser Satz lässt sich besonders gut durch das Experiment prüfen, ist jedoch oben der Einfachheit wegen unter die Axiome genommen.' (p. 109)

[25] 'nach Anbringung einer kleinen Modifikation' (p. 321).

I know, the independence of $AB \equiv BA$ is open in both Tarski's (as stated in [Gupta, 1965, p. 40]) and in Hilbert's axiom system. $AB \equiv BA$ has recently been shown dependent in a variant of Tarski's axiom system in which in the conclusion to the five-segment axiom (the version of the Side-Angle-Side triangle congruence theorem in the absence of the angle-congruence notion) the endpoints of a segment have been switched (see [Makarios, 2013]).

It is also worth noticing that in the lectures of 1898–99 we find (much as in [Greenberg, 2008, pp. 110–111]) a plane separation axiom instead of the Pasch axiom. The latter is proved from the plane separation axiom, where we also find a proof of the space separation theorem (p. 233). An even more interesting theorem that appears in all lecture notes from 1896 on (pp. 174, 257–258, 335–337), is the *Three-Chord Theorem*, which states that, if three circles pairwise intersect in two points, then the three lines joining those two points (to be referred to in the sequel as 'chords') are concurrent. Several proofs are presented, some algebraic, depending on Pappus, others by considering the three circles as belonging to three spheres. Hilbert's proofs are far from enlightening; he mentions a dependence of some proofs on the circle-continuity principle (*i.e.*, the fact that the co-ordinate field is Euclidean), and it is in general not clear what role he had in mind for this theorem. It follows from [Hartshorne, 2003], as pointed out by Marvin Greenberg [2010], that the Three–Chord Theorem, in the form stating that if two of the chords intersect, then the three chords must be concurrent, holds in plane absolute geometry (*i.e.*, in planes satisfying Hilbert's incidence, betweenness, and congruence axioms).[26] At any rate, the Three-Chord Theorem *ought* to be true, with concurrence replaced by the requirement that the three chords lie in a pencil, given that it is a universal statement that can be stated in terms of incidence and congruence or in terms of orthogonality alone[27] in Bachmann's metric planes. If we con-

[26] From [Hartshorne, 2003] it follows only that the Three-Chord Theorem holds in absolute planes that satisfy the line-circle continuity axiom (that states that a line that is incident with a point in the interior of a circle must intersect that circle). Universal statements being hereditary, the Three-Chord Theorem must hold in any submodel as well, and all absolute planes are submodels of absolute planes satisfying the line-circle continuity axiom, as every Pythagorean field has a Euclidean hull.

[27] It can be stated as

$$\bigwedge_{i<j} O_i \neq O_j \wedge \bigwedge_{1\leq i<j\leq 3} P_{ij}^1 \neq P_{ij}^2 \wedge \bigwedge_{1\leq i\leq 3, 1\leq j<k\leq 3, j=i\vee j=k, 1\leq n,m\leq 2} O_i P_{jk}^n \equiv O_i P_{jk}^m \rightarrow \kappa(P_{12}^1 P_{12}^2, P_{23}^1 P_{23}^2, P_{13}^1 P_{13}^2),$$

where $\kappa(P_{12}^1 P_{12}^2, P_{23}^1 P_{23}^2, P_{13}^1 P_{13}^2)$ stands for the fact that the composition of the reflections in the lines $P_{12}^1 P_{12}^2$, $P_{23}^1 P_{23}^2$, and $P_{13}^1 P_{13}^2$ is a line-reflection. Notice that κ does not introduce any existential quantifier, since, if a, b, c are three lines, then $\kappa(a, b, c)$ is equivalent with abc being involutory, *i.e.*, $abc \neq 1$ and $(abc)^2 = 1$

sider it as a statement of metric-Euclidean planes (see [Bachmann, 1973] for a definition of the term), then it *is* true;[28] so Hilbert's concerns regarding the Euclidean property of the coordinatizing field were misplaced, as models of metric-Euclidean planes exist with any non-quadratically closed field as coordinate field. (So there are finite models in which the Three-Chord Theorem holds, a very far cry from the requirement that the field be ordered and Euclidean, *i.e.*, that every positive element be a square.)

5. The Legacy of the *GdG*

Why is *GdG* important, why is the first edition deserving a reprint? Is it just because it has been repeated often and at regular intervals that it is a classic? *GdG* has also had its fair share of critics, beginning with Poincaré and ending with Freudenthal [1957].[29] From a social point of view, its major importance has undoubtedly been that of turning work in the foundations of geometry into a socially respectable occupation, given on the one hand that no less than Hilbert himself showed interest in it, and on the other that a school pursuing work in the foundations of geometry sprung up around Hilbert.[30] We will not try to explain the apparent historical success of *GdG*, with its fourteen German editions, given that success with an audience relies on a variety of social factors and is not always correlated with intrinsic value. What we would like to point out here is its importance *sub specie æternitatis*. Several historical studies have shown that much of the perceived pioneering quality of *GdG* is based on an ignorance of the predecessors of *GdG*. Freudenthal [1957] points out that the 'introduction of number' was the work of Karl G.C. von Staudt (in his *Geometrie der Lage* of 1847, which contains a few gaps). Toepell [1985] notes that

[28] The *power-of-a-point theorem* holds in Euclidean planes (structures with $K \times K$ as point set, with K a non-quadratically closed field, in which segment congruence $\equiv$ is defined by $(a, b)(c, d) \equiv (a', b')(c', d')$, if and only if $\|(a-c, b-d)\| = \|(a'-c', b'-d')\|$, with $\|(x, y)\| = x^2 + ky^2$, where k is a constant satisfying $-k \notin K^2$, with the 'distance' between two points defined in terms of $\|\cdot\|$ (see [Schröder, 1985]), so the radical-axis concept makes sense for two intersecting circles and the proof proceeds as in the standard Euclidean case. Since every metric-Euclidean plane can be embedded in a Euclidean plane, and the Three-Chord Theorem is a universal statement, it must hold in all metric-Euclidean planes as well.

[29] Freudenthal's criticism of the betweenness relation itself, which he deems old-fashioned, proposing the binary-order relation from algebra, and of the 'early' introduction of the order axioms [Freudenthal, 1957, p. 119] is misguided. There are alternatives for the betweenness relation, but no binary one among points, and it is Pasch's and Hilbert's 'early' introduction of the order axioms that motivated much of the very rich work in ordered geometry, surveyed in [Pambuccian, 2011].

[30] A mathematician whose name Weyl no longer recalled once told Hilbert 'You have forced us all to consider important those problems you considered important' [Weyl, 1944, p. 615].

it was Friedrich Schur who first proved ${}^{a}Papp$ with the help of congruence axioms but without the Archimedean axiom [Schur, 1899] and [Weyl, 1944], and that Hilbert's citation practice leaves much to be desired, not only in Schur's case.[31] Stroppel [2011] and Arana and Mancosu [2012] show that the independence of the Desargues theorem from the other incidence axioms is implicit in the work of Beltrami going back to 1865, and is explicit in a paper of Peano of 1894.

We believe that the significance of the *GdG* has been largely misunderstood.

Today, most mathematicians would consider it as a work in which Hilbert presented a modern axiomatization of Euclidean three-dimensional geometry, the main aim being to fill the gaps in Euclid. As such, it has been relegated to the museum of classical works, without much connection with current mathematical practice or interests. The widespread belief that Hilbert's aim was to axiomatize three-dimensional Euclidean geometry over the real numbers[32] has given 'working mathematicians' the perfect excuse to ignore the axiomatic point of view,[33] as, according to this belief, the axiomatic set-up has little bearing on the results obtained, and work in models is fully justified, as second-order logic rarely offers the option of a syntactic proof (there is no known syntactic use for the final completeness axiom V.2).

Again, in this widespread view, *GdG* could have a use as a pedagogical example of the axiomatic method, to be used in the undergraduate mathematics curriculum or in that of future high-school mathematics teachers. As such, it was found already by G.D. Birkhoff [1932] to be too demanding, and was replaced by an axiom system in which the real numbers show up on the first page, in Postulate I 'of line measure'! This tradition has been followed by most textbooks in use today, if they even pretend to provide an axiom system (many skip that task altogether, perceived as tedious,

[31] 'In his papers one encounters not infrequently utterances of pride in a beautiful or unexpected result and in his legitimate satisfaction he sometimes did not give to his predecessors on whose ideas he built all the credit they deserved' [Weyl, 1944, p. 615]. Like-minded statements can also be found in [Marchisotto and Smith, 2007], and in [Mancosu, 2010, p. 11] we read: 'In light of the importance of the work of Peano and his school on the foundations of geometry, it is quite surprising that Hilbert did not acknowledge their work in the *Foundations of Geometry*.'

[32] An entirely erroneous assumption, given that there is no completeness axiom in the *Festschrift* edition of *GdG* (it was introduced in the French translation of 1900), as pointed out in [Rowe, 2000, p. 69].

[33] As Juliette Kennedy [forthcoming] puts it, 'That mathematics is practiced in what one might call a formalism free manner has always been the case — and remains the case. Of course, no one would have thought to put it this way prior to the emergence of the foundational formal systems in the late nineteenth and early twentieth centuries.'

boring, and useless), with two major exceptions: [Greenberg, 2008] and [Hartshorne, 2000].

To think that *GdG* was about providing an axiom system for school geometry and then to complain that the axiom system is too complex is much like assuming that *Don Quixote* is children's literature, and then complaining that it is too long for its intended audience.

It has been shown by Robin Hartshorne, who has used [Hartshorne, 2000] as an undergraduate textbook at the University of California at Berkeley, that it is possible to present *GdG* to a contemporary *mathematically mature* audience — emphasizing all of the major Hilbertian themes: the continuity with Euclid's *Elements*, the avoidance of a symbolic language, the axiomatic introduction of area, the impossibility of constructions with certain sets of instruments, the continuity-free axiomatization of hyperbolic geometry in the style of [Hilbert, 1903a], as well as Dehn's solution to Hilbert's Third Problem — with complete proofs, and updated to today's language and algebraic understanding.

An alternative approach, stemming from Pieri's work and that of the Italian school, which expressed axiom systems in symbolic language, is that of Tarski, whose axiom system [Tarski and Givant, 1999] has the advantage that it is presented in a very simple language, with only points as variables, two predicates as primitive notions, and very few axioms.[34] The proofs of all theorems required to reach algebraization are provided in [Schwabhäuser, Szmielew, and Tarski, 1983]. There is no doubt that Hilbert's axiom system, with three types of variables, and a large number of predicates, was not meant to be presented as a formal system (although it has been formalized four times so far, in [Rössler, 1934; Helmer, 1935; Cassina, 1948–1949; Schwabhäuser, 1956], and one of the reasons for *GdG*'s popularity — compared to the muted response Peano's or Pieri's work has received — is precisely the natural-language presentation of the material, as pointed out in [Marchisotto and Smith, 2007, pp. 274–277]. In fact, not much has changed since March 31, 1897, when Felix Klein wrote to Mario Pieri that he would accept a survey of his results for publication in the *Mathematische Annalen* only if it was not written in formal language, since papers 'written in this symbolic language, at least in Germany, find practically no readers, but rather stumble upon rejection from the start'.[35]

[34] However, in matters of simplicity, many mathematicians consider, in the spirit of the Rumanian saying that 'The shortest road is the known road', that the simplest axiomatization is the known one, *i.e.*, one that most resembles Hilbert's. The reviewer has provided for the past 30 years various criteria for simplicity and has provided *absolutely simplest* axiom systems according to multiple criteria, only to hear that the proposed axiom systems are not at all simple. . .

[35] 'Meine allgemeine Erfahrung besteht nämlich darin, dass Arbeiten, welche in dieser Symbolik geschrieben sind, jedenfalls in Deutschland, so gut wie keine Leser finden, vielmehr von vornherein auf Ablehnung stossen.' [Luciano and Roero, 2012, p. 188]

As shown in (ii), Hilbert managed to turn a weakness — the many predicates the language has — into a strength, by asking the questions referred to earlier, many of which asked for the strength of a certain fragment of geometry, expressed in a language obtained by dropping some of the primitive notions. Even the use of the notion of angle-congruence, which had already been shown to be dispensable in [Mollerup, 1904], turned out to create both apparently intractable problems (such as the question whether the 'Side-Angle-Side' congruence axiom can be replaced by 'Side-Side-Side' or by 'Side-Angle-Angle' in the framework of Hilbert's absolute geometry (assuming the completeness axiom, 'Side-Angle-Angle' was shown in [Donnelly, 2010]) that do not exist in Tarski's axiom system, as well as axiomatizations using angle-congruence but not segment-congruence [Schütte, 1955b; Quaisser, 1973; Schaeffer, 1979; Pambuccian, 1998b].

The true value of *GdG* can be found in the unprecedented *depth* with which it treats its subject, in the stupefying magic of its models, even those that appeared at the time to be clumsy, like the model of independence for Desargues axiom in the 1898–99 lectures, which popped up in the classification of $\mathbb{R}^2$-planes with a 3-dimensional group almost eighty years later in [Betten and Ostmann, 1978] (see [Stroppel, 1998]), or the model of independence of ^{a}Des that appeared in the Festschrift edition, that is still of interest today [Anisov, 1992; Schneider and Stroppel, 2007]. The lectures only deepen that awe at Hilbert's ability to create models, and reading [Hilbert, 1903b] one has no doubt that, despite having had illustrious predecessors, by becoming the architect of barely possible worlds, Hilbert turned the foundations of geometry into something entirely different.

If we ask what the legacy of *GdG* and, more generally, of the axiomatic foundation of geometry is, we find that there are two contrasting legacies.

The first is in the written record, and it is astonishing, with more than 1300 papers and monographs devoted to the axiomatic foundation of geometry. These solve a wide range of problems and provide elementary axiomatizations for several elementary fragments of the geometries invented in the nineteenth century (the most important monographs being [Pickert, 1975; Bachmann, 1973; Benz, 1973; Schwabhäuser, Szmielew, and Tarski, 1983]. We also find, quite surprisingly, that the axiomatic foundation of geometry has remained for the most part rooted geographically in the two countries in which research in the axiomatic foundations of geometry was active in the late nineteenth century: Germany and Italy. The major exception is Poland, where research in the axiomatic foundations of geometry has been intense, and where there was no nineteenth-century contribution, the interest in this endeavor originating with Alfred Tarski. In the United States there was a short-lived interest in axiomatics, around, Eliakim H. Moore Oswald, Veblen and

Edward V. Huntington, that ended in 1930 (see [Scanlan, 1991]). Later Tarski showed considerable interest, but had very few followers in the United States. Karl Menger and his students (from Notre Dame and the Illinois Institute of Technology) provided surprisingly simple incidence-based axiom systems for plane hyperbolic geometry and for some of its fragments. Independent lines of research were pursued in the work of Michael Kallaher, James T. Smith, Marvin Greenberg, Robin Hartshorne, Michael Beeson, and Robert Knight. Mathematicians in France, Russia (or the Soviet Union), Japan, and the United Kingdom showed by and large no interest, and had a low regard for the field if they were aware of its existence.

In the collective memory of 'working mathematicians', however, there is no awareness of the axiomatic foundations of geometry as a field of research with its own challenging problems, entirely unrelated to textbook presentations of axiom systems, which was opened up by the work of Pasch, Peano, Pieri, Schur, and decisively influenced by Hilbert and the *GdG*.[36] Unless they have worked in the axiomatic foundations of geometry, mathematicians tend to be oblivious to the 130 years of research in this area that have passed since the publication of Moritz Pasch's *Vorlesungen über neuere Geometrie*, and believe that *GdG* is the crowning achievement of this axiomatizing endeavor.

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[36] There may be some awareness of the more active areas that have foundational character, such as finite geometries or buildings, but little of it is of a first-order axiomatic nature ([Shult, 2011] comes closest to an axiomatization in Hilbert's sense).

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