THE SUBSTITUTIONAL ANALYSIS OF LOGICAL CONSEQUENCE

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draft version · please don't quote
2nd June 2016

> Consequentia 'formalis' vocatur quae in omnibus terminis valet retenta forma consimili. Vel si vis expresse loqui de vi sermonis, consequentia formalis est cui omnis propositio similis in forma quae formaretur esset bona consequentia [...]
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> Iohannes Buridanus, Tractatus de Consequentiis (Hubien 1976, I.3, p.22f)

ABSTRACT

A substitutional account of logical truth and consequence is developed and defended. Roughly, a substitution instance of a sentence is defined to be the result of uniformly substituting nonlogical expressions in the sentence with expressions of the same grammatical category. In particular atomic formulae can be replaced with any formulae containing. The definition of logical truth is then as follows: A sentence is logically true iff all its substitution instances are always satisfied. Logical consequence is defined analogously. The substitutional definition of validity is put forward as a conceptual analysis of logical validity at least for sufficiently rich first-order settings. In Kreisel's squeezing argument the formal notion of substitutional validity naturally slots in to the place of informal intuitive validity.

*I am grateful to Beau Mount, Albert Visser, and Timothy Williamson for discussions about the themes of this paper.

At the origin of logic is the observation that arguments sharing certain forms never have true premisses and a false conclusion. Similarly, all sentences of certain forms are always true. Arguments and sentences of this kind are *formally* valid. From the outset logicians have been concerned with the study and systematization of these arguments, sentences and their forms. For instance, arguments in *modus Barbara* are formally valid:

> All men are mortal. Socrates is a man. Therefore Socrates is mortal.

If the terms 'man', 'Socrates' and 'mortal' are uniformly replaced with other terms, the resulting argument will never have true premisses and a false conclusion.

There are arguments in which the truth of the premisses necessarily implies the truth of the conclusion, but which are not *formally* valid, at least not by he usual standards. In particular, an argument may be *analytically* valid without being formally valid. An example is the following argument:

> John is a bachelor. Therefore John is unmarried.

Although following argument isn't analytically valid, its conclusion is necessarily implied by the premiss under common Kripkean assumptions:

> There is H_2O in the beaker. Therefore there is water in the beaker.

Arguments of this kind could be called *metaphysically* valid.

Since the analytically and metaphysically valid arguments aren't formally valid, necessary truth preservation is not a sufficient condition for formal validity. It could be objected that this depends on a certain understanding of necessity: If we understand necessity as formal or logical necessity, then necessary truth preservation is a sufficient condition for formal validity. However, tweaking the notion of necessity in this way runs the risk of becoming circular: Logical necessity cannot be analyzed again as formal validity. Hence it seems hard to avoid the conclusion that there are necessarily truth preserving arguments that aren't formally valid.

Many medieval philosopher at least from Buridan onwards were clear about the distinction between formal validity and other kinds of validity. They separated formal validity from other kinds of validity, although not all used the term 'formal' in the same way.[1] Especially in the modern period philosophers have usually relegated the analysis of analytic, metaphysical and other kinds of

[1] See (Read 2012) and (Aho and Yrjönsuuri 2009, §6.3).

‘material’ validity to philosophy of language. Logical validity has come to be understood as formal validity.[2] In what follows I identify logical with formal validity, too. There has been some resistance to this identification (see, e.g., Read 1994). The reader who prefers to understand the notion of logical validity in a less restrictive sense may take the following as an analysis of formal validity rather than full logical validity. The notion of formal validity is important enough to deserve a deeper analysis at any rate. It serves various purposes in philosophy in contexts where it has to be separated from wider conceptions of consequence such as analytic and metaphysical consequence.

THE SUBSTITUTION CRITERION

Counterexamples have always played an important role in arguing that a given argument fails to be logically valid. Traditionally, counterexamples are conceived as substitution instances. This is still the way counterexamples are often presented in introductory logic classes. A substitution instance of a given argument is obtained by substituting uniformly nonlogical (or, in a more traditional terminology, categorematic) terms with nonlogical terms of the same grammatical category. A substitution instance of an argument is a counterexample if and only if the premisses of the counterexample are true and the the conclusion is false.

The possibility of using of counterexamples for showing that an argument is not logically valid relies on the following soundness principle:

> SOUNDNESS If an argument is logically valid, it doesn’t have any counterexamples.

The converse of this principle is more problematic. It can be stated as follows, by contraposition, as the following completeness principle:

> COMPLETENESS If an argument isn’t logically valid, it does have a counterexample.

The existence of substitutional counterexamples depends on the availability of suitable substitution instances in the language. Thus it seems that this principle makes the definition of logical validity highly dependent on the language. Especially for highly restricted languages with a very limited vocabulary, as they are

[2] See Asmus and Restall (2012) for a brief historical synopsis.

often considered in mathematical logic, the completeness principle looks less plausible. The problem will be discussed in some detail below.[3]

Combining both soundness and completeness yields the substitution criterion: An argument is logically valid if and only if it has no counterexamples. In fact this principle has been used not only as a criterion but as a conceptual analysis of logical consequence.[4]

In the present paper I explore the potential of the substitutional account of logical truth and consequence for highly regimented languages, more precisely, for strong classical first-order theories such as set theory in which large parts of mathematics and, perhaps, the sciences can be developed. By making certain adjustments it may be possible to adapt the framework to certain nonclassical theories. Some remarks about this are below. I have little to say on the subtleties

[3]This was the main reason for Tarski (1936*b*) to reject a substitutional analysis, which he discussed on p. 417. He stated a substitutional definition of logical consequence as 'condition (F)' and then went on: 'It may, and it does, happen – it is not difficult to show this by considering special formalized languages – that the sentence X does not follow in the ordinary sense from the sentences of the class K although the condition (F) is satisfied. This condition ma yin fact be satisfied only because the language with which we are dealing does not possess a sufficient stock of extra-logical constants. The condition (F) could be regarded as sufficient for the sentence X to follow from the class K only if the designations of all possible objects occurred in the language in question. This assumption, however, is fictitious and can never be realized.' I will arue that this objection can be countered with a suitable substitutional definition of logical consequence.

[4]The substitution principle seems to have been used early on, at least implicitly. The quote from Buridan's *Tractatus de Consequentiis* used as motto of this paper plus parts of the following paragraph on material consequence come at least close to an endorsement of the substitution criterion as a definition of formal consequence. See (Dutilh Novaes 2012) and (Aho and Yrjönsuuri 2009, §6.3) for more details. In what follows I refer to this analysis as the substitutional definition or conception of logical consequence

According to the usual narrative, the substitutional notion of validity finally developed into the modern model-theoretic account of logical consequence via Bolzano's (1837) and Tarski's (1936*b*). Kreisel (1967) and Etchemendy (1990) both mentioned Bolzano as a precursor of Tarski's and the contemporary model-theoretic definition of logical validity. In particular, (Etchemendy 1990, p. 28ff) saw Bolzano as a proponent of a linguistic substitutional account of logical truth In footnote 2 to chapter 3 Etchemendy made some qualifications, but then says that he will 'gloss over this difference'. Ironically, Tarski (1936*b*) added later a footnote to his paper mentioning an observation by Heinrich Scholz with Bolzano's account as precursor of the definition of logical consequence advocated in Tarski's paper and not the substitutional one. Bolzano isn't the best example of a proponent of a substitutional account. In Bolzano what is substituted aren't linguistic entities but rather what Bolzano called 'Vorstellung'. This German term is usually translated as 'idea'. These *Vorstellungen* are neither linguistic nor psychological entities. So Tarski's view of Bolzano's theory may be more accurate than Etchemendy's. Medieval logicians like Buridan provide much better examples of substitutional theories.

of natural language. The reader more interested in logical consequence in natural language may take first-order languages as a test case. If the substitutional theory is successful for them, there is at least hope it can be extended to natural language.

TERMINOLOGICAL REMARK. I apply the terms 'logical validity' and 'formal validity' to sentences and arguments. When I say that a sentence is valid, this may be understood as the claim that the argument with the empty premiss set and the sentence as conclusion is valid. Occasionally the qualification 'logically' or 'formal' is omitted if there is no risk that logical validity is confused with validity in a model or other kinds of validity. For single sentences I use the term 'logical truth' as synonymous with 'logical validity'. The expression 'logical truth' suggests that it designates a special kind of truth and that logical truth is defined as truth with some extra condition. Although this understanding is in line with my own approach, it shouldn't be assumed from the outset that truth is to be defined from an absolute notion of truth that is not relative to models and an additional condition that makes it 'logical'.

PROOF-THEORETIC, MODEL-THEORETIC, AND INTUITIVE VALIDITY

In modern symbolic logic the substitutional account of logical validity has largely been superseded by the proof-theoretic and the model-theoretic analyses. According to the proof-theoretic or inferentialist conception, an argument is valid if and only if the conclusion can be derived from the premisses by using certain rules and axioms, very often, the rules of Gentzen's system of Natural Deduction. Similarly, a sentence is valid if and only if the sentence is derivable without any premisses.

Today the model-theoretic analysis is the most popular theory of logical consequence for formal languages. The substitutional and the model-theoretic analysis have a common form. Both can be stated in the following way:

> *Generalized Tarski Thesis* An argument is logically valid iff the conclusion is true under all interpretations under which its premisses are true.[5]

I call analyses of this form *semantic* to distinguish them from proof-theoretic accounts that rely neither on interpretations nor on a notion of truth. The main difference between substitutional and the model-theoretic analyses lies in

[5] Beall and Restall (2006) prefer the more general and less specific sounding term *case* over *interpretation*, when it comes to give a general form of the definition their Generalized Tarski Thesis. They credit Jeffrey (1992) with the formulation.

the notion of interpretation: On a substitutional approach, an interpretation is understood in a syntactic way as a function replacing nonlogical terms; on the model-theoretic approach an interpretation is an assignment of semantic values to the nonlogical expressions plus the specification of a domain. There is another important difference: The model-theoretic analysis of validity relies on a set-theoretic definition of truth in a model. The substitutional account requires an 'absolute' notion of truth that is not relativized to a (set-sized) model.

The substitutional account was developed into the model-theoretic definition of logical consequence mainly by Tarski, starting with his (1936*b*). The modern proper notion of truth in a model appeared only in (Tarski and Vaught 1956). For Etchemendy (1990) the the model-theoretic and substititional theories of logical consequence are both what he calls *interpretational*. Given that they both conform with the Generalized Tarski Thesis, this seems justified.

Although a lot of effort has been spent to defend inferentialist and model-theoretic accounts of logical consequence, there is a widespread belief that neither is adequate. Both approaches are not very close to the informal account of logical consequence that is often presented to students in their first logic classes. The proof-theoretic and semantic definitions also don't directly capture central features of logical consequence such as truth preservation. Neither of the two definitions obviously and directly captures our informal and intuitive notion. Consequently, some logicians have tried to come up with arguments for the claim that model-theoretic validity at least coincides with 'intuitive' validity. In particular Kreisel's (1967) squeezing argument has been employed to show that the two formal analyses coincide with the somewhat elusive 'intuitive' notion of validity. I sketch the argument for logical truth; it applies also to logical consequence in a straightforward manner.

According to the squeezing argument, provability of a sentence in a suitable deductive system ('$\vdash_{\mathrm{ND}} \phi$' in the diagram below) implies its intuitve validity. The intuitive validity of a sentence in turn implies its model-theoretic validity, because any model-theoretic counterexample refutes also the intuitive logical truth of a sentence. These two implications cannot be proved formally, because the notion of intuitive validity isn't formally defined. However, there is a formal theorem, the completeness theorem for first-order logic, that shows that the model-theoretic validity of a sentence implies its provability in the logical calculus. The implications are shown in the following diagram:

$$\begin{tikzcd} \vdash_{\mathrm{ND}} \phi \arrow[r, "\text{intuitive}", "\text{soundness}"'] & \phi \text{ is intuitively valid} \arrow[r, "\text{every countermodel}", "\text{is a counterexample}"'] & \vDash \phi \arrow[ll, bend left, "\text{completeness theorem}"] \end{tikzcd}$$

If the three implications visualized by the three arrows in the diagram hold, all three notions have the same extension. For the mathematical logicians this extensional characterization of logical validity is usually good enough. But the squeezing argument doesn't establish that any of the two formal definitions is an adequate conceptual analysis of logical validity. This situation strikes me as highly unsatisfactory: We have at least two formally precise characterizations of the extension of the notion of logical validity without having an adequate conceptual analysis.

Logical consequence isn't a pre-theoretical concept. The notion of logical consequence especially for formal languages has been honed with great rigour. It would be very strange if such a highly theoretical notion eluded any attempt to make it formally precise.

I put forward the substitutional analysis as a direct, explicit, formal, and rigorous analysis of logical consequence. The substitutional definition of logical validity, if correctly spelled out, slots directly into the place of 'intuitive validity' in Kreisel's squeezing argument, as will be shown below. The substitutional account doesn't suffer from the main problems of the proof- and model-theoretic accounts.

First, substitutional validity is closer to rough and less rigorous definitions of validity as they are given in introductory logic courses. It's also closer to how logicians over the century have specified counterexamples and established validity.

Secondly, on a substitutional account it is obvious why logical truth implies truth *simpliciter* and why logical consequence is truth preserving. On the model-theoretic account, valid arguments preserve truth in a given (set-sized) model. But it's not clear why it should also preserve simple ('absolute') truth or truth in the elusive 'intended model'. Truth-preservation is at the heart of logical validity. Any analysis of logical consequence that doesn't capture this feature in a direct way can hardly count as an adequate analysis.

Thirdly, the substitutional definition of logical consequence is not tied to set theory and its philosophy. On the model-theoretic account, interpretations are specific sets; on the substitutional account they are merely syntactic and (under certain natural assumptions) computable functions replacing expressions. Work on the model-theoretic theory of logical consequence has led philosophers to doubt that quantification 'over absolutely everything' is possible and to speculate about the indefinite extensibility of the set-theoretic universe. For the proponent of a substitutional account it is easier to avoid such speculations – at least for the sake of the theory of logical consequence. At any rate the direct link between the most complex metaphysical speculative theory hitherto, set theory, and the theory of logical consequence is severed.

The substitutional definition of logical validity seems to be a much better candidate for being a rigorous and formal conceptual analysis of 'intuitive validity'. In the next sections I make the definition of substitutional validity precise. Intuitive validity is then made formal by identifying it with substitutional validity. This will have the effect that all three implications in Kreisel's squeezing argument become formally provable implications, and we no longer have to rely on informal rigour for the argument, once the mysterious notion of intuitive validity has been replaced with the formal definition of substitutional validity.

SUBSTITUTION INSTANCES

There are two notions that need to be sharpened before the substitutional definition of validity can be made formally precise. These are the notion of a substitution instance and the notion of truth or satisfaction. I begin with the notion of a substitution instance.

Even if the logical terms are fixed, it may not be clear what counts as a suitable substitution instance. Clearly, a general term such as 'is a man' may be replaced not only with another general term but also with a complex general term such as 'is a wise philosopher with a long beard'. The argument

> All wise philosophers with a long beard are mortal. Socrates is a wise philosopher with a long beard. Therefore Socrates is mortal.

is of the same form *modus barbara* as the usual Socrates–mortality example. Similarly, we may want to allow atomic singular terms to be replaced with complex ones. Replacing proper names with definite descriptions may cause problems. This requires some care. I leave the elaboration of the formal details to another occasion.

There are more singular terms than only proper names and definite descriptions. Would we allow personal or demonstrative pronouns as substitution instances of proper names? For instance, is the following argument of the same form as the above argument in *modus Barbara*?

> All starfish live in the sea. That animal is a starfish. Therefore that animal lives in the sea.

The phrase 'that animal' can also be replaced with the single pronoun 'that', even though that may sound less idiomatic. At any rate, the resulting argument is logically valid again, as long as all occurrences of 'that animal' or just 'that' refer to the same object. Whenever both premisses are true, the conclusion is true, whatever 'that' refers to. Thus pronouns are admissible as terms that substitute singular terms.

Pronouns can also be introduced through the substitution of predicate expressions.

> All objects in the box are smaller than that (object). The pen is in the box. Therefore it is smaller than that (object).

Here 'is smaller than that' has replaced 'is mortal'. The form of the argument hasn't changed, as long as the reference of 'that' doesn't change between the premisses and the conclusion. Since occurrences of pronouns in substitution instances are going to be allowed, the definition of logical truth in natural language would require a reference to the way the pronouns are interpreted:

> A sentence is logically true iff all substitution instances are true for any reference of the pronouns.

The use of demonstrative or also personal pronouns makes it possible to formulate counterexamples or interpretations involving a singular term referring to an object for which we lack a name or definite description. In a language of first-order logic free variables can play the role of pronouns. A formula with free variables isn't true simpliciter; it's only true relative to the reference of the free variables. The reference of the free variables can be specified by variable assignments. Consequently, the definition of validity will take the following form: A sentence is logically valid iff all substitution instances are satisfied by all variable assignments. Therefore not only a notion of truth but also of satisfaction will be required.

THE CHOICE OF PRIMITIVE NOTIONS

Where does this section go?

I will provide a definition of logical validity in a base theory – with set theory being the main example – expanded with a primitive, axiomatized predicate for satisfaction. This is contrast to the usual model-theoretic analysis that doesn't require a primitive predicate for satisfaction, because satisfaction in a model can be defined in set theory alone. For the substitutional analysis set-theoretic reductionism has to be abandoned: The substitional notion of validity isn't reducible to set theory. My approach is also in contrast to other nonreductionist approaches by Field (2015) and others, who take logical validity itself to be a primitive notion that is to be axiomatized.

If no primitive notion beyond set theory itself are admitted, I cannot see how to avoid many of problems identified by Etchemendy (1990). Basic properties of logical validity become mysterious on a strongly reductive approach that excludes all notions that aren't purely mathematical. In particular, I cannot see how to obtain a definition of the logical validity of arguments that immediately entails truth preservation in valid arguments, as it's not even clear how to *state* properties such as truth preservation. For the model theorist this isn't a problem. For the philosopher it is. It has driven philosophers to postulate the existence of an elusive 'intended model' whose existence can be refuted in set theory or to doubt the possibility of quantifying over absolutely everything. The latter is already extremely difficult to state as a thesis in a nontrivial way. On a substitutional account with a primitive predicate for satisfaction there is no need for sophisticated theories about indefinite extensibility, the denial of the possibility of quantification over everything, or the belief in an elusive intended model.

Then why not go all the way and treat validity as a primitive predicate, if new undefined semantic vocabulary has to be added anyway? First, satisfaction will be required anyway for diverse areas of philosophy. For instance, in epistemology, truth is needed to formulate the fairly uncontroversial claim that the truth of a belief is a necessary condition for it to be known. Logical validity is less entrenched in various philosophical disciplines. Secondly, we cannot easily appeal to a intuitive pre-theoretic notion of logical consequence. The notion of logical is highly theoretical. Talk involving a truth predicate is much deeper entrenched in everyday language than talk about logical consequence. Thirdly, on the approach envisaged here, logical validity is definable in terms of a satisfaction predicate and set theory; conversely, however, satisfaction cannot be defined in terms of logical validity and set theory. If logical validity is treated as a primitive notion, truth or satisfaction still would have to be added as primitive notion, in order to show truth preservation and other desired properties of logical validity. That satisfaction and truth aren't definable in terms of logical validity is to be expected for purely recursion-theoretic grounds in a first-order setting: Logical validity will be extensionally equivalent to first-order provability and thus be recursively enumerable, while truth won't even be elementarily definable.

A sentence is defined to be logically valid iff all its substitution instances are satisfied by all variable assignments. The trivial substitution that maps a sentence to itself is a permissible substitution, of course. Thus, if we have the notion of truth or satisfaction relative to a substitution, we can define an 'absolute' notion of truth as truth, that is satisfaction under all variable assignments, relative to the trivial substitution. By Tarski's theorem on the undefinability of truth, such an absolute notion isn't definable. Since the substitutional account of logical validity requires a notion an absolute notion of truth, which cannot be defined, I introduce a notion of satisfaction axiomatically.

It is not necessary to axiomatize a notion of satisfaction relative to a substitution. Given an 'absolute' notion of truth in the sense of (Davidson 1973), truth relative to a substitution can be defined, as substitution is a syntactic and computable concept that is definable in weak arithmetical systems already, as long as substitution are defined in a straightforward way and the languages aren't grotesquely complicated.

insert here an account of Tarski

I suspect that the substitutional account fell out of favour because a notion of truth or satisfaction outside mathematics or set theory is required. Of course, as we have just seen, one can try to use them anyway; but then one will encounter either the problem mentioned by Tarski (or Etchemendy's *persistence* problem) or we are driven to the modern model-theoretic account that relies on truth in a model.

This, however, doesn't mean that the substitutional analysis of validity has to be abandoned. Instead of using a defined notion of truth or satisfaction, I am going to axiomatize a satisfaction predicate. Axiomatic approaches have been pursued and advocated by different authors.[6]

In this paper the axioms for satisfaction will be added to an extension of Zermelo–Fraenkel set theory possibly with urelements as the 'base theory'. The base theory can be enriched by further defined notions and further axioms and rules may be added. One could also use weaker theories as base theory such as certain arithmetical theories, but then some adjustments will be required. The base theory must contain a theory of syntax. This can be achieved in the usual way by a coding or by a direct axiomatization. Furthermore, it must contain a theory of variable assignments, that is, functions from the set of variables into arbitrary sets. With a a little extra work one can use also finite function as variable assignments, which will be necessary if one doesn't use

[6] Tarski (1935) already considered axiomatic approaches. Davidson propagated an axiomatic approach in various papers (see his 1984). Halbach (2014) provides a survey of axiomatic theories of truth for arithmetical languages.

Zermelo–Fraenkel as base theory but an arithmetical theory.

The binary satisfaction predicate $\mathrm{Sat}(x, y)$ is intended to apply to formulae x and variable assignments y. The schemata of Zermelo–Fraenkel are expanded to the full language including the satisfaction predicate. As for the axioms and rules for it, I am going to use 'compositional' axioms that match the classical logic of the base theory.

First, $\mathrm{Sat}(x, y)$ commutes with all quantifiers and connectives. Hence we have an axiom expressing that a variable assignment satisfies a formula $A \wedge B$ if and only if it satisfies A and B; a variable assignment satisfies a formula $\neg A$ if and only if it does not satisfy A; and so on for other connectives. A variable assignment satisfies a formula $\forall x\, A$ if and only if all its x-variants satisfy A. As usual, an x-variant of a variable assignment is any variable assignment that differs from it only in the value of x. The formulae and sentences here may contain the satisfaction predicate.

The language of the base theory contains various predicate and possibly also function symbols. The axioms are as expected. For instance, a variable assignment a satisfies the formula $x \in y$ if and only if $a(x) \in a(y)$, where $a(v)$ is the value of a given variable v under the variable assignment a. Similar axioms are added for all predicate symbols other than Sat. If individual constants and function symbols are present, suitable axioms have to be specified. I use the name Ω for the overall theory, comprising the axioms of the base theory, all its schemata extended to the language with Sat and the axioms for Sat.

The theory Ω resembles the usual 'Tarskian' theory of truth. It must be more or less what Davidson had in mind, with the exception that the compositional clauses are postulated also for formulae containing the satisfaction predicate. We may want to add further axioms and rules later. But for the present purposes we can proceed with these axioms.

Since also schemata are extended, Ω is properly stronger than the base theory. It's consistency cannot be proved relative to Zermelo–Fraenkel set theory. It follows, however, from the existence of a weakly inaccessible cardinal, for instance. Moreover, adding analogous axioms to reasonably behaved weaker theories such as Peano arithmetic yields consistent extensions of these theories. The truth axioms act as a reflection principle.[7] Therefore it is at least plausible to assume the consistency of Ω.

The axioms for satisfaction describe a classical notion of truth, which reflects the axioms for classical of the base theory. If a nonclassical theory were used as base theory, the axioms for justification would have to be adjusted accordingly. For instance, a theory formulated in Strong Kleene logic would require a

[7] See (Halbach 2014, sec. 22).

matching theory of truth for this logic. The substitutional approach may thus be extensible to other logics.

SUBSTITUTIONAL INTERPRETATIONS

A substitutional interpretation will be defined as a function that yields, applied to a formulae of the language (including those with the satisfaction predicate), a substitution instance.[8] The substitution will be uniform, of course. That is under a given substitutional interpretation, the same predicate symbol, for instance, will always be replaced with the same formula. By using the term *substitutional interpretation* I emphasize that role of these function is to a certain extent analogous to that of model-theoretic interpretations. But while the latter assign possible semantic values to expressions, substitutional interpretations only assign to each atomic nonlogical expression a possibly complex expression of the same syntactic category. Substitutional interpretations resemble relative interpretations as introduced by Tarski et al. (1953). Roughly, substitutional interpretations are defined like relative interpretation just without the requirement that provability is preserved.

More or less, a substitutional interpretation maps all formulae of the entire language to formulae in such a way that the logical structure is preserved. If there are no function symbols in the formula, then a substitutional interpretation is a function that replaces uniformly in every given formula each atomic formula with a possibly complex formula that contains at least the same variables as the original atomic formula; moreover, it possibly replaces every quantifier with a quantifier restricted to some fixed formula (or it may leave the quantifiers unchanged). Clashes of variables are assumed to be avoided in some of the usual ways. Individual constants are uniformly replaced with other constants or variables. If further function symbols are present, things become more complicated; I don't go into details here.

should this req be dropped?

Here I don't take a stance on the logicality of identity. We can say that a substitutional interpretation doesn't replace any occurrence of the identity symbol, so that identity is treated as a logical constant. If identity is treated as nonlogical symbol, certain well-known problems are avoided from the outset. There are some further tweaks of the definition of substitutional interpretations that will be considered in the following section.

[8] The term 'substitutional interpretation' may be somewhat misleading as it usually refers to a particular interpretation of quantifiers. This may suggest that the nonlogical vocabulary is quantified away and logical truth defined in terms of higher order quantification. This approach isn't completely unrelated, but here it's not required because in the presence of a satisfaction predicate higher-order quantifiers are not required.

The technical details of defining substitutional interpretations recursively aren't difficult. Obviously the definitions can be carried in weak system already. Assuming that there are only finitely many nonlogical symbols, substitutional interpretations are primitive recursive functions.

FREE LOGIC

In the definition of substitutional interpretations there is an issue whose direct analogue on the model-theoretic side is the question whether the empty domain should be admitted.

As explained above, a substitutional interpretation restricts all quantifiers in a formula. So, under a substitutional interpretation I, a subformula $\forall x\, F(x)$ becomes $\forall x\, (R(x) \rightarrow I(F(x)))$ and $\exists x\, F(x)$ becomes $\exists x\, (R(x) \wedge I(F(x)))$. The restricting formula $R(x)$ corresponds to the domain of a model-theoretic interpretation, that is, of a model.

I have not ruled out relativizing formulae such as $P(x) \wedge \neg P(x)$ that do not apply to anything. This corresponds to an empty domain on the model-theoretic side. Of course, empty domains are not admitted in the standard semantics of classical first-order logic. If this is to be emulated on the substitutional approach, the definition of substitutional interpretations can be changed as follows: The substitutional interpretation $I(F)$ with relativizing formula $R(x)$ of a formula F containing, for example, exactly the individual constants a and b is preceded by the expression $\exists x R(x) \wedge R(a) \wedge R(b) \rightarrow \ldots$. The formula $\exists x R(x)$ expresses, model-theoretically speaking, that the domain isn't empty, while $R(a)$ ensures that the constant a denotes an object in the domain.

move outside Sat

The exclusion of the empty domain in models strikes me as a philosophically not very convincing oddity that is accepted mainly for convenience. One difficulty with free logic is the definition of truth from satisfaction. As Schneider (1958) noticed, if a sentence is defined to be true iff it's satisfied by all variable assignments, then all sentences will be true in the model with the empty domain, because there are no variable assignments over the empty domain. If a sentence is defined to be true iff it's satisfied by at least one variable assignment, no sentence is true. The problems can be solved. Williamson (1999) discussed some workarounds; but for most purposes it's much more convenient just to exclude the empty domain. Usually logicians consider theories that imply existential claims. Thus the empty domain is excluded by nonlogical axioms and the model with the empty domain is irrelevant for the analysis of these theories.

For the analysis of logical validity, however, there are good reason to retain the empty domain, if validity is analyzed in the model-theoretic way, even if this means that the definitions of satisfaction and truth become more cumbersome. On the substitutional account the analogous move is much more straightforward. On the contrary, free logic looks much more natural, because the antecedent including $\exists x R(x)$ of each interpretation can be omitted. However, there are still decisions to be made as far as individual constants are concerned. If negative free logic is chosen, then to the substitutional interpretation of an atomic formula $F(a)$ the conjunct $R(a)$ should be adjoined, where $R(x)$ is the relativizing formula of the substitutional interpretation. This ensures that formulae with constants not denoting objects satisfying $R(x)$ are not satisfied by any variable assignment.

If a free logic approach is chosen, one may think about adapting the logic of the base theory and, in particular, the treatment of individual constants. The necessary modifications are straightforward. If the language of the base theory doesn't contain any individual constants, there is no need for any modifications, as the base theory will contain existential claims.

Here I don't discuss the pros and cons of the different varieties of free logic. Many arguments that have been made in favour and against certain varieties in terms of models can be rephrased in terms of substitutional interpretations. If a language without constants is considered and no modifications are made to the definition of substitutional interpretations in the previous section, then sentences of the form $\exists x(P(x) \vee \neg P(x))$ will not come out as logically valid under the definitions of logical truth in the next section. The notion of validity will be that of free logic. The most straightforward definition of validity on the substitutional account yields a logic without any ontological commitment. I take this to be an advantage of the substitutional analysis. No additional trickery is required as on the model-theoretic account.

SUBSTITUTIONAL DEFINITIONS OF LOGICAL TRUTH & CONSEQUENCE

Using the axiomatized notion of truth and the defined notion of a substitutional interpretation, logical truth and consequence can now be defined. A formula A is satisfied under a substitutional interpretation I and variable assignment iff the substitutional interpretation $I(A)$ of that formula is satisfied under that variable assignment. A formula is logically valid iff it is satisfied under all substitutional interpretations and variable assignments. A sentence A follows logically from a premiss set Γ iff A is satisfied under all substitutional interpretations and all variable assignments under which all formulae in Γ are satisfied.

The analysis expounded here may look like a hybrid between a substitutional and model-theoretic accounts. Logical validity is defined in terms of satisfaction and a substitutional interpretation may replace an individual constants not only with an individual constant, but also with a free variable. Hence the usual worry about objects not named by any constant or other singular term is alleviated. A given constant may be replaced with a free variable and then that variable can be assigned any object via the variable assignments.

In contrast to individual constants, predicate expressions cannot be replaced with (second-order) variables. Substitutional interpretations substitute predicate symbols with formulae. However, such a substitution may introduce more first-order variables. For instance, the atomic formula Px with the unary predicate symbol P has $Px \wedge Py$ as a substitution instance. Thus, a substitutional interpretation maps the formula with only the variable x free to one that has both x and y free. This is in line with the policy on constants: It should not matter whether we have a name for an object or not. For instance, a substitutional interpretation may map Px to $Px \wedge Pa$; if the constant a isn't available, one can still achieve the same effect by interpreting Px by $Px \wedge Py$ and choose a variable assignment that assign y the object denoted by the constant a. As has been explained above, this corresponds to the well-established use of pronouns in arguments and counterexamples in traditional logic.

Although substitutional interpretations may introduce new free variables and logical validity isn't defined in terms of truth but satisfaction, the definitions are still substitutional. As pointed out before, using pronouns in counterexamples isn't a problem for a substitutional definition of validity; analogously, the use of free variables shouldn't be seen as a departure from the substitutional approach.

The definitions of logical truth and consequence yield some welcome benefits. With these definitions it is trivial that logical validity implies truth. If a sentence is logically valid, it is satisfied under all substitutional interpretations and variable assignments. The identity function, that is, the function that maps every formula to itself is a substitutional interpretation. Hence, if a sentence is valid on the substitutional definition, it is satisfied under all variable assignments, that is, it is true. Similarly, it can be established that logical consequence preserves truth. omit?

If $\mathrm{Val}(x)$ expresses the above notion of validity, then the claim that logical validity implies truth becomes $\forall x\,(\mathrm{Val}(x) \to \forall a\,\mathrm{Sat}(x,a))$, where $\forall a$ expresses quantification over all variable assignments (the restriction of the quantifier $\forall x$ to sentences isn't necessary). This principle isn't even expressible on the model-theoretic account, because there is absolute set-theoretically definable satisfaction predicate Sat. Thus one might protest that $\forall x\,(\mathrm{Val}(x) \to \forall a\,\mathrm{Sat}(x,a))$ doesn't express what is meant by saying that logical validity implies truth. What is really meant, so one might claim, is the schema $\mathrm{Val}([A]) \to A$ for all sentences A, where $[A]$ is some canonical name for the sentence A. This schema, however, strikes me as too weak to express the claim that logical validity implies truth. Unlike the universally quantified principle, it's not clear how to negate the schema. so we cannot even state that validity doesn't imply truth on the schematic account. But one can still ask whether the schema $\mathrm{Val}([A]) \to A$ is provable. In fact it is, but perhaps, the critical reader might feel, for the wrong reasons.[9]

PERSISTENCE

The main reason for abandoning attempts to analyze logical validity via substitution has been the worry that on the substitutional account an invalid argument or a sentence may declared valid.[10] On the substitutional account, a counterexample is a suitable substitution instance. The set of substitution instances is limited by the language. But validity shouldn't depend on the vocabulary at our disposal. Persistence is a requirement of the analysis of validity: A sentence that is logically valid, must not become invalid if the language is expanded and more substitution instances become available.

If very confined the languages are considered and strong restriction on possible substitutions are imposed, the problem is obvious. For instance, if individual constants may only be replaced with individual constants and the language contains only the two constants a and b that happen to denote the same object, then $Pa \to Pb$ will be valid according to such a restrictive substitutional theory of validity.

Such restrictions to expressively weak languages, however, are not a real problems for the substitutional analysis of logical validity. Restricting substitution instances to expressions of a very restricted language is in conflict with

[9] The proof is somewhat convoluted. First it can be shown that $\mathrm{Val}([A])$ implies truth in all set-theoretic models and hence provability in (free) predicate logic by the formalized completeness theorem. The theory of satisfaction is essentially reflexive and thus proves the local reflection principle for any finite subtheory, and thus for logic. A proof via the satisfaction predicate isn't feasible, even if the theory of satisfaction is (consistently) strengthened. The schema $\forall a\, \mathrm{Sat}([A], a) \to A$ is inconsistent, as Sat commutes with negation. Only the rules that allows to proceed from a proof of $\forall a\, \mathrm{Sat}([A], a)$ to A can consistently be added.

[10] This historical claim needs to be substantiated. Tarski (1936*b*) rejected a version of the substitutional account characterized by his condition (F). Etchemendy discussed the persistence problem in various places in his (1990). I suspect, the reason why logicians in earlier times haven't been bothered so much by the problem is that they didn't fix a language and weren't afraid to use demonstrative pronouns and similar devices, so that any object could be designated perhaps not by a proper name or description but at least by a pronoun.

the universality of logic. If a sentence is logically true, then all substitution instances should be true, not only those in some toy language. This doesn't mean that we have to consider all possible expansions of the language of the base theory. All substitution instances in our language have to be true. If we use set theory, then it must be possible to substitute set-theoretic vocabulary. As will become obvious, what matters is not so much that we have many expressions for substituting at our disposal, but rather that expressively and deductively rich parts of our language can be substitution instances. In particular, being able to substitute formulae with only the symbol for set-theoretic membership as nonlogical symbol for a predicate expression will go a long way.

By comparison, the model-theoretic analysis is far from being safe from similar problems. First of all truth in all models doesn't include truth in the intended model. So there is a substitutional interpretation without a model-theoretic counterpart. This problem has been remarked upon by many authors. But there are also other worries for the model-theoretic analysis. When doing model theory it's not completely clear in which theory exactly model theory is developed. But usually model theory is carried out in mathematics or, more narrowly, in set theory. Non-mathematical objects and sets with non-mathematical objects as urelements are not usually considered. The persistence worry is here that counterexamples with urelements are missing. The worry can be alleviated, as McGee (1992*a*) argued, if it is assumed that for any structure there is an isomorphic structure that is a pure set, that is, a set whose transitive closure doesn't contain anything but sets. At any rate , also the mode-theoretic accounts requires arguments showing that persistence doesn't fail.

The main objection against the substitutional account from the proponents of the model-theoretic analysis will be that there are set-theoretic models that don't have a substitutional counterpart. There are only countably many substitution instances of a given sentence; but the number of set-theoretic models isn't limited by any cardinality.

I now show that this objection doesn't pose a problem for the substitutional analysis, if the above substitutional definitions of validity are employed. The substitutional interpretation of a sentence can contain more free variables than the original sentence. For the sentence to be valid the substitutional interpretation must be satisfied by all variable assignments. By varying the variable assignments there is more leeway for constructing counterexamples: Counterexamples are not only obtained by substituting the nonlogical vocabulary but also by varying the assignment of objects to the variables. In fact, on the substitutional approach, a counterexample can be understood as a pair of a substitutional interpretation and a variable assignment. Since there is no limit on the cardinality of variable assignments, there is also no such limit on the cardinality of counterexamples. Hence, the above worry that there are only countably many substitutional counterexamples doesn't apply.

The observation below establishes that, if a sentence is logically valid on the substitutional account, it is also valid on the usual model-theoretic account. More precisely the following claim can be proved in the theory Ω: *If there is a set-theoretic model in which which A is false, there is a substitutional interpretation and a variable assignment under which the sentence A isn't true.* The proof can also be adapted to the analogous claim for arguments.

Proof. I outline how to prove the result in Ω. Assume that there is a set-theoretic model $M \nvDash A$. Let $x_1, \ldots, x_n$ be the finite list of all variables occurring in A. Let m and a be variables not in this list. The substitutional interpretation I is now defined in the following way. An atomic subformula in A can be of the form Rx_1x_2. The interpretation I maps Rx_1x_2 to the formula $m \vDash Rx_1x_2(a(x_1/[x_1], x_2/[x_2]))$. The latter formula is still formulated in the language of set theory, and thus in the language of the base theory; it expresses that the formula Rx_1x_1 is holds in the model m under the variable assignment a but with the value for the variable 'x_1' changed into x_1 and the value for the variable 'x_2' changed into x_2. Thus in $m \vDash Rx_1x_2(a(x_1/[x_1], x_2/[x_2]))$ the free variables are x_1, x_2, m and a. It is a formula in the language of set theory. The predicate symbol R could be the satisfaction predicate; but then the overall formula is still in the language of set theory, because R is only mentioned and not used. Other atomic formulae are dealt with in an analogous way. The interpretation restricts all quantifiers in A to the domain of m. The interpretation $I(A)$ of the sentence A will contain exactly two free variables m and a. Validity in m commutes with all connectives and quantifiers in A, because all quantifiers are restricted to the domain of m. Thus $I(A)$ is equivalent in set theory and Ω to $m \vDash A(a)$. Now let b be a variable assignment that assigns M to m and some object to a (the value of a would only matter, if A contained free variables). Then $m \vDash A(a)$ and $\mathrm{Sat}([m \vDash A(a)], b)$ are equivalent in Ω. Hence Ω proves that $M \nvDash A$ implies that there is a substitutional interpretation I and a variable assignment b such that $\neg\mathrm{Sat}(I(A), b)$. This concludes the proof.

The proof can be adapted to various situations. In particular, it applies to the various variations of the definition of substitutional interpretations and the related free logics. In this case the systems of Natural Deduction and the model-theoretic definitions must be adjusted.

The proof establishes more than the mere claim that there is a substitutional interpretation and a variable assignment under which the sentence A isn't true, if there is a set-theoretic model in which which A is false. Each set-theoretic model maps to a different pair of a substitutional interpretation and a variable assignment. So the set of potential countermodels isn't impoverished by passing from the model-theoretic to the substitutional theory.

Conversely, in contrast, there are pairs of a substitutional interpretation and a variable assignment that don't have a model as counterpart. In particular, any pair with the identity substitutional interpretation that maps any formula to itself (with possible adjustments from the section on free logic) doesn't correspond to any model-theoretic interpretation, because the domain of a model is always a set.

add here formalized completeness

The formally rigorous squeezing argument

For a suitable definition of substitutional interpretations (to exclude free logic), as sketched above, substitutional validity, on the one hand, and proof-theoretic and model-theoretic validity on the other coincide.

Kreisel's squeezing argument applies to substitutional validity. The notion of substitutional validity isn't purely set-theoretic, but it's nevertheless formally rigorous. This formally rigorous notion can now replace the elusive notion of intuitive validity in the squeezing argument:

$$\begin{tikzcd} \vdash_{ND} \phi \arrow[r, "\text{soundness}", "\text{theorem}"'] & \phi \text{ is substitutionally valid} \arrow[r, "\text{persistence}", "\text{theorem}"'] & \vDash \phi \arrow[ll, bend left, "\text{completeness theorem}"] \end{tikzcd}$$

The proof that provability in Natural Deduction implies validity can now be formalized in Ω. It is a simple inductive proof that makes the informal argument in Kreisel's original version of the proof explicit and formal. To establish that substitutional validity implies validity in all set-theoretic models, the persistence proof in the preceding section is used. Again this is a proof in the formal theory Ω. Hence all implications in the diagram are formally rigorous.

The squeezing argument still serves its purpose: It shows that by concentrating on set-theoretic countermodels nothing is lost. There are substitutional countermodels that don't have no direct set-theoretic counterpart. These are the substitutional countermodels without a restricting formula that defines a set.[11] In particular, on the model-theoretic account the 'intended interpretation'

[11] Participants of the FilMat16 in Chieti suggested the use of set-theoretic reflection principles to show a direct equivalence of substitutional and model-theoretic validity without going through the Gödel completeness theorem. This strategy has its limitation if the equivalence is to be shown for arguments with infinite premiss sets.

cannot be used as a countermodel. The formally rigorous squeezing model shows that the mode-theoretic definition still succeeds in classifying sentences and arguments correctly as logically valid or not. Thus the original worry about the model-theoretic analysis of validity is alleviated. The absence of the 'intended interpretation' from the set-theoretic interpretations, that is, models doesn't render the model-theoretic analysis extensionally inadequate.

Worries about the intended interpretation do not affect the substitutional account as there the inteded interpretation is among the permissible interpretations. In fact it's the trivial substitutional interpretation that maps every sentences to itself.

THE INTENDED INTERPRETATION

probably this should come earlier

If logical truth is understood as truth under all interpretations, then it's an oddity of the model-theoretic account that there isn't a special interpretation of a given sentence that allows one to understand the sentence at face value. I suppose that many philosophers and logicians have felt the need for such an 'intended' interpretation. Of course there are other motives for talking about the 'set-theoretic universe'. Most set-theorists will happily talk about the set-theoretic universe, the cumulative hierarchy, and so on. Of course, there are ways to make this precise in a class theory and declare the universe to be a proper class about which we can reason in our class theory. However, as is well known, this move will not help. Of course we can start with a class theory as base theory. But then the intended interpretation would have to contain all sets and, in addition, proper classes. Such objects can be talked about in a theory about superclasses and a type theory over set theory can be set up. This doesn't look very appealing. It certainly isn't standard practice; and one will never reach the elusive intended interpretation.

A kind of dual approach is to resort to the indefinite extensibility of the set-theoretic universe. The approach is hard to describe consistently. Very roughly speaking, it is denied that quantifiers range over all sets. The universe a quantifier ranges over, turn out to be a set V_α, that is, a level in the cumulative set-theoretic universe; but this set is then contained in another level and so on. Of course this is very similar to postulating classes, superclasses and further levels.

It's not only that set theory doesn't prove the existence of the set-theoretic universe. Standard Zermelo–Fraenkel set theory *refutes* the existence of a universal set. Here I don't intend to go too deeply into the philosophy of set theory. But it strikes me as very strange that the theory of logical consequence should push us into the most extravagant metaphysical speculations about elusive universes and intended interpretations that slip away like Berkeley's elusive subject as soon as we think we have snatched a glimpsed of it.

Some theory textbook contain explanations how to understand what looks like talk about proper classes as a *façon de parler*. For doing set theory a universe isn't needed as an object. We may prefer to use the symbols V, L, or On, as if they were constants for proper classes; but their use can easily be explained away.[12]

In the case of logical truth and consequence this isn't so easy. There we need to quantify over all interpretations in the definitions of logical validity. The intended interpretation ought to be included in the range of the quantifier over interpretations. Hence proper classes cannot so easily be explained away as a *façon de parler*.

At least in the present context, the need for the elusive object arises from the model-theoretic understanding of 'intended interpretation'. The use of model theory for the analysis of logical validity (and some other philosophical purposes) creates the need for an object that contains all the objects that can be quantified over. There are good reasons to resist the temptation of postulating the existence of such a universal object, avoid the talk about indefinite extensibility, the problems of absolutely unrestricted quantification, set-sized universes as philosophical 'models' of the full universe, and so on.

In the substitutional account of logical validity, the need for all this controversial and suspect theorizing disappears, at least as far as logical validity is concerned. The intended interpretation becomes something that is very simple. It is far from being mysterious or elusive. It simply is the function that maps every formula of the language to itself. There is no longer a motive for trying to form a universal object. Since the intended interpretation in the sense of the substitutional account can be used in a completely unproblematic way, the proof that logical truth implies simple truth becomes obvious, just as it should be.

NECESSITY AND LOGICAL VALIDITY

The connection between modality and logical validity has become a widely discussed topic in itself that has spilled over into logic textbooks that define logical validity in terms of possible worlds. This is misleading. Metaphysical

[12] There are different ways to do this. Obviously, a formula such as $x \in On$ ist shorthand for a formula expressing that x is an ordinal. Even what looks like limited quantification over classes can be eliminated. See, for instance, (Lévy 1979, Appendix X).

validity, that is, necessary consequence isn't *logical* consequence, as explained at the beginning of this paper. On the model-theoretic and the substitutional account of logical validity and necessity are disentangled. This is an advantage of both accounts.

Of course, there are historical precedents for a close connection between logical validity and necessity, starting with Aristotle. But it's far from clear what kind of necessity early authors had in mind. Logicians have managed to disentangle logic from necessity. Necessity is needed neither for the proof-theoretic nor for the model-theoretic nor the substitutional analysis. I think it's better to leave them disentangled.

There are, however, arguments supposed to show that the notion of necessity cannot be so easily purged from the theory of logical validity. Some philosophers, including Etchemendy (1990), have been worried that on certain accounts of validity – accounts Etchemendy calls 'interpretational' – whether a sentence is valid may depend on what the world is like. For instance, if there were only finitely many objects, then the negation of the conjunction of all axioms of Robinson's arithmetic Q or any finitely axiomatized theory with only infinite models would be logically valid.

This worry is hard to understand. Whether sentences or arguments are valid or not depends on certain mathematical assumptions. By 'depend' I mean that, if we change our base theory, other sentences and arguments could be declared valid or invalid. This kind of argument could be made against many formal and mathematical results. What would happen to theorems in number theory if all the prime number bigger than 101 didn't exist? Would the cut elimination theorem fail if only sentences up to a length of 10^{20} existed? Would $\forall x \forall y\ x = y$ be logically true if only one object existed? I find these questions futile. If one thinks that mathematical and formal objects exist only contingently, then these mind-boggling questions may become relevant. If it is assumed that mathematical objects and, in particular, sets exist necessarily the problem disappears (see McGee 1992*b*). This is the case on the substitutional as well as the model-theoretic account.

For the sake of this section, I just assume that mathematical objects exist by necessity without defending the view. Then, for instance, the claim that the negation of conjunction of all axioms of Robinson's arithmetic Q isn't logically true holds by necessity. Claims about logical validity have a status comparable to mathematical truths: They hold by necessity. However, this should not lead one to believe that they are themselves logical truths. A confusion about this point may also be to some extend the source of the so-called paradoxes of validity. That there are no such paradoxes has been shown by Ketland (2012) and Cook (2014).

Even though pure mathematical objects exist by necessity, it doesn't mean that mathematical theorems are logically true. The sentence claiming that there are at least two objects is necessarily true, but it isn't logically true, even if it's assumed that identity is a logical constant. This substitutional interpretations can restrict quantifiers. Of course, some authors, including Williamson (2000), defend the logical validity of the claim that at least two exist. Such an approach is compatible with the substitutional theory of logical validity expounded in this paper by redefining substitutional interpretations without quantifier restrictions. An analogous move isn't possible on the model-theoretic account, because one cannot simply admit proper classes as domains of models. Nevertheless I prefer to reject the logical validity of the claim that there are at least two objects. Logical validity should be purely formal and independent of the subject matter whether the subject matter exists by necessity or not.

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