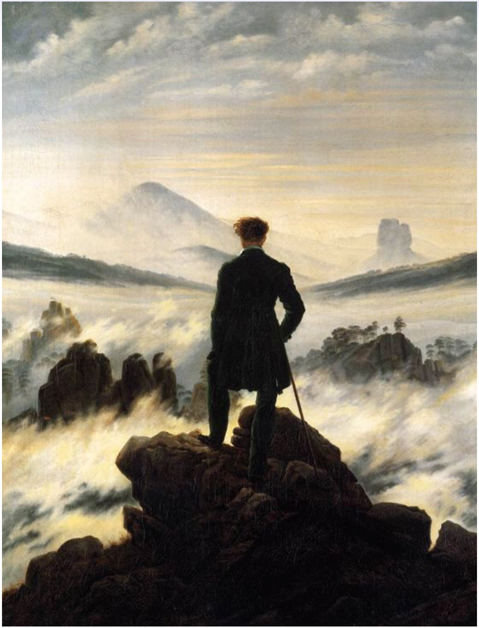

Axioms as Procedure and Infinity as Method, both in Ancient Greek Geometry and Modern Set Theory

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Axiomatics arising *in* mathematical practice occurred in ancient Greek geometry and modern set theory.

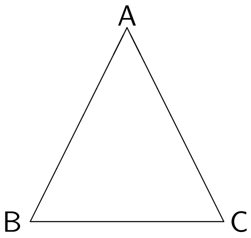
For both, it is corroborated, by looking at the history and practice, that axioms (and definitions) serve to warrant procedures (e.g. constructions) and so to regulate infinity as method.

Axioms in practice are thus less involved with mathematical truth or metaphysical existence than with the instrumental properties of concepts and the regulation of procedures.

Greek Geometry

Thales (c. 624-546 BCE): propositions, concepts, and equality, e.g.

- A diameter of a circle bisects the circle.
- The base angles of an isosceles triangle are equal.



Pappus' Proof

Mathematikoi (early 5th century BCE):

- *number* as a collection of units; odd vs. even numbers, figurative numbers.
- ratio and proportion.
- The arithmetical mean $\frac{a+b}{2}$, the geometric mean $\sqrt{ab}$, the harmonic mean $\frac{2ab}{a+b}$.
- Pythagorean Theorem $a^2 + b^2 = c^2$.

Latter 5th Century BCE, Problems and Deductions

Hippocrates of Chios (c. 479-410 BCE):

- Worked on Cube Duplication, *reduced* it to the construction of two mean proportionals (given a, b to construct x, y such that $a : x :: x : y :: y : b$).
- Worked on Circle Quadrature, constructed quadrature of luneles.
- Mastery of *Elements* I (Basic), III (Circles), VI (Similarity).

(c. 420 BCE) Incommensurability of the side and diagonal.

Theodorus of Cyrene (c. 465-398 BCE):

- Worked on Incommensurable Magnitudes, e.g. $\sqrt{n}$ for $n < 17$.
- Mastery of *Elements* II (Geometric Algebra).

Early 4th Century BCE, Plato's Academy

Archytas of Tarentum (c. 429-350 BCE):

- Cube Duplication via intersection of three solids to find two mean proportionals.
- Worked on Incommensurable Magnitudes.

Theatetus (c. 417-369 BCE):

- Worked on Incommensurable Magnitudes with anthyphairesis.
- Mastery of *Elements* VII (Number Theory), XIII (Construction of Regular Solids), some X (Irrational Lines).

Eudoxus of Cnidus (c. 408-355 BCE):

- Cube Duplication via curved lines generated through mechanical motion.
- Proportion for Incommensurable Magnitudes, *Elements* V (Theory of Proportion) and completion of X (Irrational LInes).
- Method of Exhaustion.

Latter 4th Century and Early 3rd Century BCE

- Further efforts at Circle Quadrature.
- Angle Trisection.
- Analysis of conic sections.
- Reductive analysis of problems back to *Elements* II context.
- Euclid's (lost) *Porisms*, *Data*, and *Elements*.

Euclid's *Elements* (c. 300 BCE)

The *Elements* codified the more elementary geometry established through straightedge-compass constructions. It was carefully crafted to establish propositions in a deliberately contrived deductive sequence that exhibits the minimal assumptions.

The *Elements* consists of *propositions* which can be classified as of two kinds:

Problems: to construct a configuration for which a specified property is to hold.

Theorems: With a construction of a figure given by hypothesis, to confirm that a specified property holds.

In the main, the proofs are *synthetic*, moving forward from givens, but can be evidently seen as having been arrived at through *analysis*, working back from the conclusion. Euclid's *Data* has a close symbiosis for *Elements* I-VI, presenting a catalogue of steps that can be used in analysis.

As a bridge between analysis and synthesis, *diorisms* in propositions state necessary conditions for a construction to be successful.

Book I is a remarkable achievement, getting to the Pythagorean Theorem without Geometric Analysis (Book II) or Similarity (Book VI). It states with basic definitions, and:

Postulates:

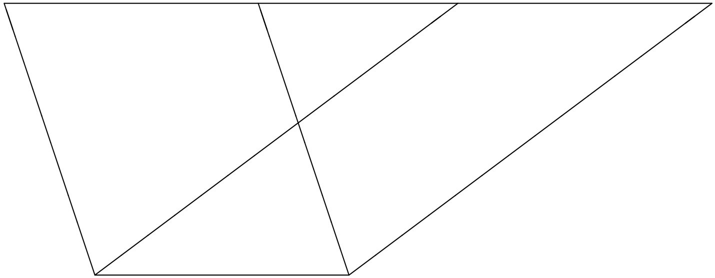
- Let it have been postulated to draw a straight-line from any point to any point.
- And to produce a finite straight-line continuously in a straight-line.
- And to draw a circle with any center and radius.
- And that all right-angles are equal to one another.
- And that if a straight-line falling across two straight-lines makes internal angles on the same side less than two right-angles, then the two straight-lines, being produced indefinitely, meet on that side on which are the angles less than the two right-angles.

Common Notions:

- Things equal to the same thing are also equal to one another.
- And if equal things are added to equal things then the wholes are equal.
- And if equal things are subtracted from equal things then the remainders are equal.
- And things coinciding with one another are equal to one another.
- And the whole is greater than the part.

The 2nd and 3rd Common Notions are used for the first time in the following. It is the first proposition asserting area equality beyond congruence, and the first clear example of analysis vs. synthesis.

Elements I 35: Parallelograms which are on the same base and between the same parallels are equal to each other.



- Book III, Definition 5 (Equality of Circles):
  Equal circles are those whose diameters are equal, or whose radii are equal.
- Book V, Definition 4 ("Archimedes' Axiom"):
  Those magnitudes are said to have a ratio with respect to one anther which, being multiplied are capable of exceeding one another.
- Book V, Definition 5 (Eudoxus' Equality of Ratios):
  Magnitudes are said to be in the same ratio, the first to the second, and the third to the fourth, when equal multiples of the first and the third either both exceed, are both equal to, or are both less than equal multiples of the second and the fourth, respectively, being taken in corresponding order, according to any kind of multiplication whatever.

Archimedes (c. 287-212 BCE)

- Took an earlier, Eudoxan approach.
- Reduced Circle Quadrature to *rectifying* the circumference.
- Circle Quadrature via spirals—depends on recognized assumptions.
- Problems via conic sections, e.g. divide a sphere according to a given ratio.
- Angle trisection via neusis and spirals.

Archimedes' *On the Sphere and the Cylinder I*

Postulates:

- That among lines which have the same limits the straight line is the smallest.
- And, among the other lines (if, being in a plane, they have the same limits), that such lines are unequal, when they are both concave in the same direction and either one of them is whole contained by the other and by the straight line having the same limits as itself, or some is contained, and some it has as common, and the contained is smaller.
- And similarly, that among surfaces, too, which have the same limits (if they have the limits in a plane) the plane is the smallest.

- And that among the other surfaces that also have the same limits (if the limits are in a plane) such surfaces are unequal, when they are both concave in the same direction, and either one is wholly contained by the other surface and by the plane which has the same limits as itself or some is contained, and some it has as common, and the contained is smaller.
- ("Archimedes' Axiom") Further, that among unequal lines, as well as unequal surfaces and unequal solids, the greater exceeds the smaller by such a difference that is capable, added itself to itself, of exceeding everything set forth (of those which are in a ratio to one another).

Doxography and Historiography

- Plato
- Aristotle
- Proclus
- Zeuthen
- Hilbert
- Unguru Controversy

Set Theory

At first a foundational investigation involving the infinite as method, and then developing into an investigation of well-foundedness through (transfinite) recursion.

Cantor:

- Established results about infinite cardinality.
- Established transfinite numbers and advanced transfinite recursive definition.
- Continuum Hypothesis $2^{\aleph_0} = \aleph_1$: a construction principle.

Zermelo's 1908 Axioms:

- Extensionality (two sets are equal if they contain exactly the same members): an instrumental definition.
- Pairs $\{x, y\}$, Union $\bigcup x$, and Power Set $\mathcal{P}x$: constructions.
- Infinity (There is an *inductive set*, i.e. a set X such that $\emptyset \in X$, and if $x \in X$, then $x \cup \{x\} \in X$.): warrants Mathematical Induction, the principle that characterizes the least inductive set, ω.
- Separation (For any set X and property φ, $\{x \in X \mid \varphi\}$ is a set): construction through property.
- Choice (Every set X has a *choice function*, i.e. a function f such that for any nonempty $y \in X$, $f(y) \in y$): a construction principle, best seen as such through the equivalent Zorn's Lemma, which warrants the recursive construction-definition of maximal elements.

The Fraenkel(-Von Neumann) Replacement Axiom, 1922 (for any set X and term $t(\cdot)$, $\{t(i) \mid i \in X\}$ is a set): a construction principle, one that warrants transfinite recursion through all the ordinals.

Zermelo's Foundation Axiom, 1930: well-foundedness and cumulative hierarchy, warrants transfinite recursions for establishing properties for *all* sets.

Zermelo himself, and decades later e.g. Scott and Shoenfield, motivated all the axioms in terms of the cumulative hierarchy.

Large Cardinal Axioms, 1960-1990

Reference: Kanamori, *The Higher Infinite*, 2012.

- Measurable Cardinals: well-founded ultrapowers
- Supercompact cardinals, etc.: ultrapowers and embeddings
- Small Large Cardinals: combinatorics
- Strong and Woodin Cardinals: extenders and inner models
- Determinacy: games, extenders, and inner models

Forcing Axioms, 1970-1990

- Martin's Axiom (If P is a c.c.c notion of forcing and D a collection of fewer than $2^{\aleph_0}$ dense subsets of P, then there is a D-generic filter): a construction principle in terms of forcing, generalizing the Continuum Hypothesis.
- Proper Forcing Axiom (If P is a proper notion of forcing and D a collection of $\aleph_1$ dense subsets of P, then there is a D-generic filter): codifies Shelah's proper forcing, and involves large cardinals and determinacy.
- Martin's Maximum (If P is a stationary-set-preserving notion of forcing and D is a collection of $\aleph_1$ dense subsets of P, then there is a D-generic filter): maximal form of Martin's Axiom having to do with supercompact cardinals.

Weaker Set Theories

- Kripke-Platek Set Theory, 1966: "characterizes" transfinite recursion as method.

- Mathias' Prov, cf. *Provident sets and rudimentary forcing*, Fundamenta Mathematicae 56(3-60), 2015: "characterizes" forcing as method.
  Mathias developed a theory of *rudimentary* recursion and with it, the *provident* sets. A set is provident exactly when it satisfies the finite axioms Prov. In Prov + Infinity, a parsimonious development of forcing can be carried out.

Set-theoretic Truth and Existence

With the plethora of models for set theory, *conceptions* of the *multiverse* (the models of set theory taken collectively together) emerged, some with axiomatizations, which have raised new propositions about set-theoretic truth and existence.

Mulitverse Conceptions:

- Realist (Hamkins): These models all "exist", each exhibiting a different concept of set.
- Non-realist (Shelah): Carnapian, pluralist. Models are constructed in practice.
- Set-generic (Woodin, Steel): Multiverse of generic extensions set in a meta-framework for getting at "set-theoretic truth".
- Vertical (Zermelo): Cumulative hierarchy of realist power set and extensible height.
- Hyperuniverse (S. Friedman): Multiverse of all countable models for set theory, to be investigated in terms of maximality principles for getting at "set-theoretic truth".