# Chapter 10
# Absoluteness and the Skolem Paradox

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## 1 Introduction

When seen in the "correct" light, the contradictions of set theory are by no means disastrous, but instructive and fruitful. For instance, the antinomies of Russell and Burali-Forti live on in the systems of axiomatised set theory in the guise of established theorems. Zermelo used the Russell-Zermelo argument to prove that every set possesses a subset which cannot be an element of that set, and from which it follows that there can be no universal set ((Zermelo, 1908b, pp. 264–265), p. 203 of the English translation), and the essentials of the Burali-Forti argument can be used to prove that there is no ordinary *set* of all (von Neumann) ordinals.[1] The fact that these contradictions reappear as theorems in set theory is not surprising given that the reasoning involved is (or can be turned into) set-theoretic reasoning, and that we always have the choice of treating the derivation of contradictions as arguments in *reductio* form, choosing one premise as responsible for the contradiction. Indeed, much of the early discussion of the arguments was concerned with

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[1] The Burali-Forti argument is explicitly used in this way in (von Neumann, 1928, p. 721), though the argument was clearly known to von Neumann much earlier, since all the essentials are present in von Neumann's new account of the ordinals from the early 1920s (see von Neumann, 1923), and this form of the Burali-Forti argument is explicitly mentioned by von Neumann in a letter to Zermelo from August 1923. (See Meschkowski, 1966, pp. 271–273.) Zermelo had given the definition of the "von Neumann" ordinals by 1915, and possibly as early as 1913. In a lecture course in Göttingen in the Summer Semester of 1920 entitled "Probleme der mathematischen Logik," Hilbert shows (pp. 15–16) how the Burali-Forti Paradox can be reproduced in the framework based on Zermelo's definition: if $W$ is the set of all ordinals, then $W$ would itself be an ordinal according to the definition, and must therefore be a member of itself, contradicting one of the central theorems about these ordinals. The lectures can be found in Chapter 2 of Ewald and Sieg (2011). Despite Zermelo's precedence here, von Neumann is still the real discoverer of the von Neumann ordinals, for he was the first to give a complete presentation of the relevant theoretical material, and in particular to recognise the importance of the Axiom of Replacement.

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assessing which of the premises used was in fact so reduced. Once absorbed into set-theoretic frameworks designed especially to avoid the known contradictions, the paradoxes give rise to arguments which reveal something deep and interesting about the existence of sets and the structure of the universe of sets itself. These systems *analyse* what is exposed by the antinomies; unsurprisingly, the stronger the system, the more refined the analysis tends to be. The most striking example is von Neumann's system, which allows that there *is* a greatest (von Neumann) ordinal; it is just that this ordinal cannot be an ordinary set, and thus cannot give rise to an even greater ordinal. Moreover, in this setting, the universal set, the Russell set, and the set of all ordinals are all equipollent, all maximally "big," and all equally "too big" to be sets. (See Hallett, 1984, pp. 288–295.)

The situation with the "semantic" antinomies is somewhat different. They, too, give rise to concrete, mathematical results. For instance, the Liar Paradox, developed within a consistent theory which allows for the right representability, yields, instead of a contradiction, Tarski's Theorem on the undefinability of the truth-predicate for that theory. However, the form of the argument is specific neither to set theory nor to set-theoretic reasoning, but applies to a wide range of languages/theories.

Nevertheless, there is a famous semantic paradox which *is* specific to set theory, namely the Skolem Paradox which goes back to (Skolem, 1923). Unlike the antinomies, this does not arise from a pre-axiomatic, set-theoretic contradiction; indeed, it is a "paradox" only made possible by the first-order axiomatic representation of sets, which Skolem (in the paper mentioned) was the first to present.[2] Skolem's argument shows that, while the axioms can prove the existence of sets with increasing infinite cardinality, yet the central concepts concerning the different infinite cardinalities must be "relative inside axiomatic set theory," since one can find an interpretation of the axioms in the natural numbers. Thus, the axiom system, if consistent, has an interpretation which cannot be the one intended, since it is based on only countably many objects. As Skolem himself pointed out, there is nothing directly contradictory about this. The argument can be sharpened (see below, p. 200f.), and crucially then internalised, to produce a contradiction, which can then be used as the basis of *reductio* arguments of great import, as Gödel first showed. What this internalisation demonstrates is that Skolem's relativity is reflected within the theory itself (i.e., without any reference to "exterior" models) in the notion of *absoluteness*. In particular, it can be used to show that, while the natural numbers are absolute, the continuum (and the power set operation generally) is non-absolute, where "absolute" can be given a precise theoretical sense.

The present paper is concerned with this internalisation. There are two things in particular which I wish to bring out. The first is historical. The non-absoluteness of the continuum focuses on a feature of extensions which was first isolated (and

[2] For discussions of Skolem's assessment of the argument, and his changing views on its consequences, see (Benacerraf, 1985; George, 1985). For more general discussion of the consequences of the argument, see (Wright, 1985).

objected to) by Poincaré. Poincaré identified this as an instability in the extensions of certain sets, and saw this as problematic, since it conflicts with his generational view of sets. Poincaré's views are set out in the first section of the paper. Axiomatic set theory, as was clear from Zermelo's initial axiomatisation, rejects Poincaré's generational view, and Poincaré's association of set existence with definability. (In large part, Poincaré was reacting to the view of sets put forward by Zermelo.) However, the internalisation of the Skolem Paradox serves to refocus both Poincaré's and Skolem's reservations. According to set theory, there is nothing unstable or relative about the continuum. Nevertheless, I will suggest that the involvement of the crucial notion of absoluteness in both Gödel's and Cohen's arguments for the consistency and independence of the GCH indicates rather a conceptual weakness in the fundamental notion of cardinality. This was something glimpsed by Skolem himself in 1923.

## 2 "Impredicative" Extensions

The isolation of non-absolute sets is foreshadowed in both Russell's and Poincaré's diagnoses of the antinomies.[3] The term "impredicative" was originally used by Russell in a quite general way to refer to properties whose extension cannot be sets and in this sense the properties used to specify the paradoxical sets are demonstrably "impredicative": see (Russell, 1907, p. 34).[4] This same paper (marked "Received November 24th 1905.—Read December 14th 1905"), written before Russell had fixed on a solution, and before he had stated the VCP, contains the following striking passage:

> ... there are what we may call *self-reproductive* processes and classes. That is, there are some properties such that, given any class of terms all having such a property, we can always define a new term also having the property in question. Hence we can never collect *all* the terms having the said property into a whole; because, whenever we hope we have them all, the collection we have immediately proceeds to generate a new term also having the said property. (Op. cit., 36.)

Russell describes clearly this kind of "self-reproduction" found in the traditional antinomies.

Suppose we call $V$ a temporary universe. Let $\psi$ be some property. It seems that we can define a set $u$ as $\{x : x \in V \wedge \psi(x)\}$. Either $u \in V$ or $u \notin V$. But suppose that $\psi$ is the property involved in the Russell contradiction, and suppose further that $u \in V$. Then we have immediately that $u \in u \leftrightarrow u \notin u$, i.e., the Russell contradiction. Suppose now that $\psi$ is the property $Ord(x)$. We can easily show that

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[3] That Poincaré was essentially concerned with non-absoluteness is a suggestion I first heard propounded in a lecture given by Wilfrid Hodges in London in 1974 or 1975. Fuller treatments of Russell's and Poincaré's views can be found in, e.g., (Goldfarb, 1988, 1989).

[4] In this paper, and elsewhere, Russell uses the term "proper class" where we would now use the term "set".

$u = \{x : x \in V \wedge \psi(x)\}$ is an ordinal (a von Neumann ordinal), and thus is not a member of itself. Thus, if $u \in V$, we would have, on the contrary, that $u$ *does* belong to itself, again the well-known contradiction. Hence, if we accept the definitions of $u$ as good, the conclusion must be that, with respect to the properties $\psi$ involved in these two paradoxes, $V$ is not the *full* universe; it is indeed only temporary, for there are perfectly good sets, like our $u$'s, which cannot possibly belong to it. We could say that what Russell shows is that there are $\psi$ such that for any temporary universe $V$ it must be the case that $\{x\colon \psi(x)\}^V \neq \{x\colon \psi(x)\}$.[5] Put more in Russell's way, whenever we call a halt to the "process" of collecting together the $\psi$, we find that there is at least one more $\psi$ that is yet to be accounted for. If, in addition, we think of *producing* sets, then these $\psi$ are naturally described, in Russell's phrase, as "self-reproductive" properties.

Russell came to the conclusion, as did Poincaré, that what characterises these collections is a certain circularity in their specification, and both believed that what is needed is some adherence to (a form of) the *vicious circle principle* (VCP) in order to avoid this. Unlike Poincaré, Russell held that what is required is an alteration to "current logical assumptions," and came to believe that these alterations must be guided by the VCP.[6] But what concerns us here is not so much the solution to the problem, but rather its diagnosis. It is clear that the idea of the VCP owes much to the analysis above. Here, for example, is Russell's statement from 1908:

> Thus all our contradictions have in common the assumption of a totality such that, if it were legitimate, it would at once be enlarged by new members defined in terms of itself.
>
> This leads us to the rule: "Whatever involves all of a collection cannot be one of the collection"; or, conversely: "If, provided a certain collection had a total, it would have members only definable in terms of that total, then the said collection has no total."†
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> †When I say that a collection has no total, I mean that statements about *all* its members are nonsense.[7]

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[5] The analogy with Zermelo's argument that the universe is not a set is clear. Russell is close to isolating the notion of non-absoluteness, though his condition is stronger; a property $\psi$ is absolute when it is possible to find at least one $V$ such that $\{x\colon \psi(x)\}^V \neq \{x\colon \psi(x)\}$, not that this necessarily holds for all $V$.

[6] For the remark about "current logical assumptions," see (Russell, 1907, p. 37). Gödel points out (Gödel, 1944, p. 135) that there are actually three distinct formulations of the VCP relied on in Russell's writings. For Gödel's discussion of these, see op. cit., pp. 455ff. Goldfarb suggests in (Goldfarb, 1989) that for Russell these formulations may be more intimately connected than Gödel's discussion allows.

[7] Russell (1908, p. 225). The relation between the "self-reproductive" properties isolated by Russell and the VCP was well summed up by Gödel:

> I mean in particular the vicious circle principle, which forbids a certain kind of "circularity" which is made responsible for the paradoxes. The fallacy in these, so it is contended, consists in the circumstance that one defines (or tacitly assumes) totalities, whose existence would entail the existence of certain new elements of the same totality, namely elements definable only in terms of the whole totality. This led to the formulation of a principle which says that no totality can contain members definable only in terms of this totality [vicious circle principle]. ((Gödel, 1944, p. 133). The square brackets are in the original.)

Russell came to focus more on the direction taken by his footnote, though I want to focus here on the formulation "If, provided a certain collection had a total, it would have members only definable in terms of that total, then the said collection has no total," for this is the closest to Poincaré's view of sets, to which I now turn.

Poincaré takes as his starting point neither Russell's antinomy nor Burali-Forti's, but rather Richard's, an antinomy which we now classify as "semantic," involving essentially a linguistic component. Consider the set $E$ of all real decimals which can be defined in a finite number of words; $E$ is obviously denumerable. The contradiction now goes, quoting Poincaré, as follows:

> Suppose the enumeration [of $E$] effected, and let us define a number N in the following manner. If the $n^{\text{th}}$ decimal of the $n^{\text{th}}$ number of the aggregate $E$ is
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> 0, 1, 2, 3, 4, 5, 6, 7, 8, or 9
>
> the $n^{\text{th}}$ decimal of N will be
>
> 1, 2, 3, 4, 5, 6, 7, 8, 9, or 0
>
> As we see, N is not equal to the $n^{\text{th}}$ number of $E$, and since $n$ is any number whatsoever, N does not belong to $E$; and yet N should belong to this aggregate, since we have defined it in a finite number of words.[8]

Poincaré endorses what he says is Richard's own solution to the antinomy, namely:

> $E$ is the aggregate of *all* the numbers that can be defined in a finite number of words, *without introducing the notion of the aggregate E itself*, otherwise the definition of $E$ would contain a vicious circle; we cannot define $E$ by the aggregate $E$ itself. Now it is true that we have defined N by a finite number of words, but only with the help of the notion of the aggregate $E$, and that is the reason why N does not form a part of $E$. ((Poincaré, 1906, p. 307); see also (Poincaré, 1908, pp. 206–207), pp. 480–481 and 190 respectively of the English translations.)

The concentration on "vicious circles" is then taken to be the way to avoid all the antinomies:

> But the same explanation serves for the other antinomies, and in particular for that of Burali-Forti. . . .
>
> Thus *the definitions that must be regarded as non-predicative* [in Russell's sense] *are those which contain a vicious circle*. The above examples show sufficiently clearly what I mean by this. ((Poincaré, 1906, p. 307); see also (Poincaré, 1908, p. 207), pp. 481 and 190 respectively of the English translations.)

The connection to the formulation of Russell's VCP which I picked out (i.e., "If, provided a certain collection had a total, etc.") is clearest in Poincaré's phrase "we cannot define $E$ by the aggregate $E$ itself." From this, Poincaré questions the boundaries of the sets picked out:

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[8] Poincaré (1906, pp. 304–305); see also (Poincaré, 1908, p. 202), pp. 478 and 185–186 respectively of the English translations. Recall that Poincaré, in this section of his paper ("Les Antinomies Cantoriennes") is explicitly discussing (Russell, 1907).

> A "non-predicative class" [thus, one whose definition suffers from a vicious circle] is not an empty class, but a class with unsettled [*indécise*] boundaries. ((Poincaré, 1906, p. 310); see also (Poincaré, 1908, pp. 206–207), pp. 481–482 and 191 respectively of the English translations.)

This is elaborated in a discussion from 1909:

> There is no actual infinity, and when we speak of an infinite collection, we mean a collection to which we can add new elements continually (similar to a subscription list which will never close, waiting for new subscribers). For the classification could only be closed properly when the list was closed; for every time that one adds new elements to the collection, one modifies it. It is therefore possible that the relation between this collection and the elements already classified is modified; and since it is according to this relation that these elements have been arranged in this or that drawer, it can happen that, once the relation is modified, the elements are no longer in the right drawer, and it will be necessary to move them. While there are new elements to be introduced, it is to be feared that the work [of classification] will have to begin all over again; and it will never happen that there will not be new elements to introduce. The classification will never be finished.
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> From this there emerges a distinction between two types of classification applicable to the elements of infinite collections, *predicative* classifications, which cannot be disrupted [*bouleversées*] by the introduction of new elements, and *non-predicative* classifications for which the introduction of new elements requires constant reshaping [*remanier*]. ((Poincaré, 1909, p. 463), or (Poincaré, 1913a, pp. 9–10), p. 47 of the English translation.)

Let us examine a little more closely what exactly worries Poincaré.

In the 1909 essay, Poincaré says that the problem concerns collections which are "mutable [*muables*]" (the collections with "unsettled boundaries" from 1906), while the passage quoted above focuses on "disrupted classifications." Though connected, the two are not exactly the same. What does he mean by a "mutable" collection? By this, Poincaré means a collection whose extension does not remain fixed over time, so that in using a term for it, its reference may vary in the course of an argument, thus rendering the conclusion of the argument uncertain. That this is one thing which worries him is made clear at the beginning of the 1909 paper. Suppose we want to use simple syllogistic reasoning to conclude that, since two soldiers are in the same regiment, they are in the same division. The proof only works, Poincaré says, as long as the soldiers stay in the same regiment; the conclusion might well be false if *in the course of the argument* one of the soldiers is transferred to a different regiment. As Poincaré says:

> What is then the condition under which the rules of this logic are valid? It is that the classification adopted is *immutable* [*immuable*]. ((Poincaré, 1909, p. 461), or (Poincaré, 1913a, p. 8), p. 45 of the English translation.)

The relevant classification here is the separation of the collection of all soldiers into regiments, and a soldier shifting regiments changes the classification.

Poincaré is clearly assuming that the argument about the soldiers is not timeless. In most ordinary cases, we rely on general assumptions about short-term stability in the macro-world which more or less guarantees that the conclusion will apply to the situation stated at the beginning of the argument. In any case, with ordinary arguments, mutability will usually depend on contingencies which are outside the scope of the argument; this is enough to suggest caution, for bad luck could dictate that the assignment of soldiers to regiments might change during the course of an argument. But it is different with the antinomies. These (for Poincaré) present cases where the classification *must* change, not because of external contingencies, but by the very nature of the argument itself.

Take for instance the Richard antinomy. We start with a term "$E$," which refers to the list of all definitions. Then we construct a definition $d$ (of N), which we have reason to think cannot be in the list denoted by $E$. But $d$ is a definition, so it must be in the list $E$, since, by assumption, $E$ is the list of all definitions. Poincaré's point, it seems, is that the extension of the term "$E$" is no longer the same at the *end* of the Richard argument as it was at the beginning, i.e., it is of necessity "muable." At the start, $d$ was not in the extension of $E$; indeed, the intention on which $d$ rests is predicated on the assumption that it is *not* in $E$. Yet, once $d$ has been framed, it seems that we are forced to admit that it must belong to $E$. Hence, the extension of $E$ is no longer the same, and the reference of "$E$" is uncertain. The extendability of $E$ rests in an essential way on $d$: it seems that we cannot formulate $d$ *without* incurring a change in the extension of $E$. In other words, mutability is *intrinsic* to the argument, and this is what forces a contradiction. We are forced to the conclusion that we are dealing with properties which, in Russell's phrase, are "self-reproductive." What seems to underline the ambiguity is that $d$ includes a universal quantifier over $E$. If the extension of $E$ changes, $d$ will no longer mean the same *after* it is formulated (with the extended $E$) as it was intended to mean at the moment it was formulated. Consequently, the truth-value of $\forall x \in E\, \psi(x)$ will in general vary as the extension of $E$ varies. Thus, we have a claim similar to Russell's in the passage quoted above (see p. 192), namely that part of the worry concerns the scope of universal quantification.

Poincaré's conclusion is that the whole line of argument is doomed from the beginning, and that it was not just dangerous to employ the term for $E$, but illegitimate, since it can never fulfill one of his conditions on the correct use of names, that its reference be 'entirely determined,' to use Poincaré's words from a later paper.[9]

Poincaré applies this analysis not only to the antinomies, but to the Cantor diagonal argument as well. Here, we start with a countable list $E_R$ of real numbers, and then define a number N in much the same way as above, i.e., so that it is immediately clear that it cannot be in the list $E_R$. Although the definition of N contains a reference to the list $E_R$, there is, it seems, no obvious circularity of the kind that worries Poincaré, since *there is no reason whatever* to think that N should itself be in the list $E_R$. However, we *can* generate a contradiction by assuming that $E_R$ consists of *all* real numbers, and thus that it must include N. In this case, there *is* an indirect reference in the definition of N to N itself, for the extension of $E_R$ is assumed to include *all* reals. But what does this contradiction show? One way to read it is as showing just that the assumption that $E_R$ contains all real numbers is demonstrably

[9] See (Poincaré, 1912, p. 8), or (Poincaré, 1913a, p. 90), p. 71 of the English translation.

false. In this case, there is not the stretching of the extension of $E_R$ that Poincaré perceives in the argument in Richard's antinomy; nothing in the proof forces us to conclude that the extension of $E_R$ is not fixed. But Poincaré rejects this reading of the argument. Why?

The answer is that Poincaré assumes what might be called a "genetic" or "generational" view of mathematical objects, according to which the stock of mathematical objects varies over time, just as the composition of a regiment will change over time. The clearest confirmation of this is to be found in Poincaré's paper from 1912 where he distinguishes between two points of view, the view that sets are picked out by what he calls "comprehension," and the view that they are created by what he calls "extension." Poincaré introduces the term "Pragmatist," and then says:

> The Pragmatists adopt the point of view of extension, and the Cantorians the point of view of comprehension. For finite sets, the distinction can only be of interest to formal logicians, but it appears to us much more profound where infinite sets are concerned. Adopting the extensional viewpoint, a collection is constituted by the successive addition of new members. By combining old objects, we can construct new objects, and then, with these, newer objects; if the collection is infinite, it is because there is no reason for stopping.
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> On the other hand, from the point of view of comprehension we start from a collection where there are pre-existent objects, objects which appear to us indistinct at first, but some of which we finally recognise because we attach labels to them and arrange them in drawers. But the objects precede the labelling, and the objects will exist, even though there may not be a curator to classify them.[10]

Poincaré adopts the "Pragmatist" or "extensional" view against the "comprehension view." According to the comprehension view, objects pre-exist, and definitions then only serve the function of selecting certain of them. Indeed, this is the core of Zermelo's defence of impredicative definitions against Poincaré's objections. Poincaré's extensional view, however, is most certainly a "generational" view, according to which objects are created in stages, as the reference to "successive additions" makes clear. But according to what, for Poincaré, does the "successive addition of objects" take place? As the passage above says, this occurs through construction, or by constructive definition. For this view, definition does not serve the same function it does for the comprehension view; correct definition is taken itself to be the criterion of existence, and does not itself rely on a supposition of existence independent of the definitional prescription. What this means is that objects are indissolubly tied to the definitions of them.

This has certain strong consequences for Poincaré's "Pragmatist." First:

> For example, the Pragmatists admit only those objects which can be defined in a finite number of words. Possible definitions, being expressible in sentences, can always be enumerated with the ordinary numbers from one to infinity. ((Poincaré, 1912, p. 5), or (Poincaré, 1913a, p. 88), p. 68 of the English translation.)

[10] Poincaré (1912, p. 4, 1913a, pp. 87–88), pp. 67–68 of the English translation. The mention of drawers in this passage recalls what Poincaré has to say in his earlier paper from 3 years before; see p. 194 above. It is very likely that Poincaré distinguishes between the "extensional" and the "comprehension" views precisely because Zermelo hints at such a distinction in the section of (Zermelo, 1908a, pp. 117–118) which replies to Poincaré. See also (Hallett, 2010, pp. 109–112).

A little later in the same paper, Poincaré says:

> And why do the Pragmatists refuse to admit objects which are not capable of a definition in a finite number of words? It is because they consider that an object only exists when it is thought, and that one will not be able to conceive of an object independently of a thinking subject. ((Poincaré, 1912, p. 9), or (Poincaré, 1913a, p. 94), p. 72 of the English translation.)

Here there is an implicit assumption that the thought of the cognising subject is necessarily tied to its linguistic expression. On its own, this is innocuous enough, and certainly would not rule out the construction of a power set $P(\omega)$ from $\omega$ in one step. But Poincaré adopts the further view that a set cannot be said to exist *before* its members have been shown to exist, thus that the existence of the members of a set must be logically prior to the existence of the set itself. (This surely must be one reason for the designation "extensional" for the position Poincaré supports.) This leaves it quite vague as to what "logical priority" amounts to, and as to how exactly the stages of existence are to be marked. But the fact that existence must be tied to linguistic specification reveals the central aspect of Poincaré's "logical priority" view. As Goldfarb points out

> Since it is first the specification that legitimizes the entity specified, that specification can in no way depend on the existence of the entity. Therefore, the ranges of the quantifiers in the specification cannot include the entity. (Goldfarb, 1989, p. 25.)

Thus, if we claim that $v \in u$, then we must be able to define $v$ with a specification $\varphi_v$ which does not involve reference to the set $u$, either directly or through a quantifier. This, plus the priority thesis, entails that the set itself can be referred to neither in its own specification nor in the specification of any of its members.

This position can be summarised in the following two theses:

> *The Linguistic Specification Thesis* (LST): A mathematical object (in particular, a set) cannot be said to exist until there is a finite linguistic specification (definition) of it.

This is combined with:

> *The Individual Specification Thesis* (IST): Before a set $A$ can be said to exist, we must be in possession of specifications of all of its potential members; needless to say, to avoid circularity, these specifications cannot make reference, either direct or indirect, to the set $A$.

It follows that a set $A$ cannot be defined by direct or indirect reference to itself, and neither can it contain members which are defined by reference to $A$.

The following passage from (Poincaré, 1912) shows that what we have just described is a fair reflection of Poincaré's "Pragmatist" position. Poincaré considers the impredicative definition of an object $X$ whereby first $X$ is defined using its relation to all the members of a collection $G$, and where it is then asserted that $X$ is itself a member of $G$. He says:

> For the Pragmatists, such a definition implies a vicious circle. One cannot define $X$ without being acquainted with [*sans connaître*] all the objects of the *genre* $G$, and consequently without being acquainted with $X$ which is one of these individuals. The Cantorians do not admit this. The *genre* $G$ is given to us; consequently we are acquainted with all these

individuals, and definition has only the aim of discerning [*discerner*] among them those which have the relation stated to their fellows.

No, reply their adversaries, the knowledge of the *genre* does not allow you to know all the individuals; it only gives you the possibility of constructing as many of them as you wish. They exist only after they have been constructed, that is to say after they have been *defined*. $X$ only exists by its definition which has sense only if one knows in advance all the individuals of $G$, and in particular $X$. ((Poincaré, 1912, p. 7), or (Poincaré, 1913a, p. 91), pp. 70–71 of the English translation.)

Note that Poincaré (the Pragmatist) quite explicitly rejects the position, attributed to the "Cantorians," that specification of a set is enough to give us "acquaintance" with its members.

The use of impredicative definitions is then tied to the worry about the instability of extensions, for Poincaré says the following:

Why do the Pragmatists make this objection [to the impredicative definition of $X$]? It is because the *genre* $G$ appears to them as a collection which is capable of increasing indefinitely whenever new individuals are constructed which possess the appropriate characteristics. Thus $G$ can never be posited *ne varieteur* as the Cantorians posit it, for we are never sure that $G$ will not become $G'$ in the light of new annexations. ((Poincaré, 1912, p. 9), or (Poincaré, 1913a, p. 93), p. 72 of the English translation.)

And in a later paper, he makes the same explicit connection:

. . . Richard's law of correspondence lacks a property which, borrowing a term from the English philosophers, one can call "predicative." (With Russell, from whom I borrow the word, a definition of two concepts $A$ and $A'$ is non-predicative when $A$ appears in the definition of $A'$ and conversely.) I understand by this the following: Every law of correspondence assumes a definite classification. I call a correspondence predicative when the classification on which it rests is predicative. However I call a classification predicative when it is not altered by the introduction of new elements. With Richard's classification this, however, is not the case. Rather, the introduction of the law of correspondence alters the division into sentences which have a meaning and those which have none. What is meant here by the word "predicative" is best illustrated by an example. When I arrange a set of objects into various boxes, then two things can happen. Either the objects already arranged are finally in place. Or, when I arrange a new object, the existing ones, or a least a part of them, must be taken out and rearranged. In the first case I call the classification predicative, and in the second non-predicative. ((Poincaré, 1910, p. 47), p. 1073 of the translation.)

This analysis is what is directly applied to the Cantor diagonal argument:

For example, the Pragmatists admit only objects which can be defined in a finite number of words; . . . [W]hy then do we say that the power of the continuum is not that of the whole numbers? Yes, being given all the points of space which we know how to define with a finite number of words, we know how to imagine a law, itself expressible in a finite number of words, which makes them correspond to the sequence of whole numbers. But now consider the sentences in which the notion of this law of correspondence figures. A few moments ago, these sentences had no sense since this law had not yet been invented, and they could not serve to define points of space. Now they have acquired a sense, and they will allow us to define new points of space. But these new points of space will not find any place in the classification adopted, and this will compel us to upset it [*la bouleverser*]. And it is this which we wish to say, according to the Pragmatists, when we say that the power of the continuum is not that of the whole numbers. We wish to say that it is impossible to establish between these two sets a law of correspondence which is secured against this sort

> of disruption [*bouleversement*]; ... ((Poincaré, 1912, p. 5), or (Poincaré, 1913a, pp. 88–89), p. 68 of the English translation. See also (Poincaré, 1909, pp. 463–464), (Poincaré, 1913a, p. 10), p. 47–48 of the English translation.)

When we are given a list of real numbers, a finite string $S$ of words specifies this list, giving a sequence $E_R$ of reals. This brings into existence a new mathematical object, namely this sequence $E_R$. Sentences which previously contained the string $S$ were strictly speaking meaningless, and therefore could not possibly define real numbers. Once it is recognised that $E_R$ exists, then such sentences have a perfectly good meaning, and some of them may well define real numbers. But no such real numbers (as so defined via $E_R$) were available for classification when $E_R$ itself was being defined, and so cannot be assumed to figure in $E_R$. Indeed, the argument shows how to define (with direct reference to $E_R$) a real number N which *cannot* be in the list. Once $E_R$ is available, the classification of the real numbers currently available thus has to begin anew, and will necessarily lead (so the argument) to a list different from $E_R$.

This is really the same as the analysis of the Richard antinomy. Unsurprisingly, Poincaré sees the two arguments as simply two aspects of the same process. We can always specify a list of definitions of real numbers, and this specification will then turn certain sentences which were hitherto meaningless into perfectly good definitions of reals. This will necessitate a new list, and so on *ad infinitum*. Indeed, the analysis reconciles an apparent contradiction between the two arguments, a contradiction which can be put starkly by stating that Richard demonstrates that there are only countably many numbers (since there are only countably many possible definitions), while Cantor shows that there are uncountably many:

> In this there lies the solution of the apparent contradiction between *Cantor* and *Richard*. Let $M_0$ be the set of all whole numbers, and $M_1$ the set of all points of our line definable by finite sentences of our table after a first run through the list, and let $G_1$ be the law coordinating both sets. Using this law, a new set $M_2$ is definable and is thus added. For $M_1 + M_2$ there is a new law $G_2$, using which there arises a new set $M_3$, and so on. *Richard*'s proof teaches us that, wherever I break off the process, then there's a corresponding law, while *Cantor* proves that the process can always be continued arbitrarily far. There is thus no contradiction between these conclusions. ((Poincaré, 1910, pp. 46–47), p. 1073 of the translation.)

The assimilation of, say, the Burali-Forti antinomy to Richard's may seem a little odd to modern eyes; since (Ramsey, 1926), it has been usual to point to a semantic element behind Richard's antinomy which is not present in Burali-Forti's or Russell's. But the reason for Poincaré's assimilation is clear. Not only does he adopt a generational view of sets, whereby the existence of the elements of a set precede that of the set itself, he makes the existence and the constitution of a set dependent on its mode of definition. Given this, the Burali-Forti antinomy does look somewhat similar, as we saw via Russell's "self-reproductive processes." The set $u$ of all ordinals could not exist without being defined (LST), and the definition uses a predicate $Ord(x)$ which $u$ itself would satisfy, showing that $u$ must be a member of $u$, thus violating IST. The central principles here, LST and IST, are decisively rejected by modern set theory.[11] Does this mean that all the important aspects of Poincaré's analysis must similarly be rejected? The answer is clearly no, as will be shown in what follows.

## 3 The Paradox Internalised

Before proceeding to the internalisation of the Skolem argument, let us quickly run through the latter's main steps.

Consider the first-order Zermelo-Fraenkel system (ZFC) with a standard list of axioms, including the Axiom of Choice (AC), and assume that it is consistent. Since the theory is written in a countable language $\mathcal{L}_{\mathrm{ZF}}$, with "$\in$" as its only non-logical predicate, then by the Completeness Theorem, it must have a countable model $\mathcal{M} = <M, E>$, where $M$ is the $\mathcal{M}$-domain, and $E$ is $\mathcal{M}$-"membership." $M$ has in it an element $m_c$, which is the interpretation in $\mathcal{M}$ of ZFC's term "$c$" for the continuum. Moreover, ZFC can prove the statement "$\neg C(c)$," where "$C(x)$" is the usual $\mathcal{L}_{\mathrm{ZF}}$ formula expressing countability. $\mathcal{M}$, being a model of ZF, therefore says that the continuum $m_c$ is uncountable. But it is clear that $m_c$ has only *countably* many $\mathcal{M}$-members, i.e., the collection $\{x : x \in M \;\&\; xEm_c\}$ must be countable, since $M$ has only countably many members. Therefore, it seems, ZFC can have models in which the continuum is both countable and uncountable.

Although this conclusion is somewhat odd (a "paradox"), it is no antinomy. For one thing, the $\mathcal{L}_{\mathrm{ZF}}$ statement "$\neg C(c)$" simply says that there is no surjective function whose domain is $\omega$ and whose range is $c$. And since $\mathcal{M}$ is a model of set theory, this must mean that there is no object in $M$ which plays the role of such a function in $\mathcal{M}$, despite the countability of $m_c$. In other words, $m_c$ is uncountable in just the way that $c$ is.[12]

One might think that what this result shows is simply that first-order set theory has unintended interpretations, if any at all, and this we now regard as nothing unusual. For one thing, so much of the work using interpretations, going back to Hilbert's construction of models for various geometries, and including Henkin's construction of a model for any consistent theory from its syntax, shows us that there is a sense in which the "material" of an interpretation is often quite irrelevant; what matters, rather, is the way it is *arranged*. The non-standard models of first-order arithmetic underline this; it is not the nature, or quantity, of the elements themselves which matters, but they way they are structured. Is this also the case with the Skolem Paradox? For example, it is quite possible that $m_c$ itself is *uncountable*, i.e., has uncountably many *real* members, although contains only countably

[11] Something like the IST is pursued in Martin-Löf's constructive type theory; see e.g., (Nordström et al., 1990, p. 27).

[12] This resolution of the paradox was pointed out by Skolem in his original paper, (Skolem, 1923, p. 223).

many $E$-elements. Is the oddity just due to the fact that the relation $E$ of the model $\mathfrak{M}$ is itself just non-standard? Short reflection shows that this sanguine reaction is misplaced.

Suppose that ZFC has a model $\mathcal{V}$, with domain $V$, which we regard as an *intended* model, and thus has the real set membership over it, thus $\in$. By the Downward Löwenheim-Skolem Theorem, there exists a countable sub-structure $\mathcal{V}' =< V', \in'>$ of $\mathcal{V}$, where $V' \subseteq V$, and $\in'$ is just $\in$ itself restricted to $V'$; this sub-structure satisfies precisely the same $\mathcal{L}_{\text{ZF}}$ sentences as does $\mathcal{V}$, and its membership relation really is the same, intended set-membership relation. Although itself countable, this $V'$ might still contain uncountable members. $\mathcal{V}'$ satisfies the Axiom of Extensionality, so $V'$ must be extensional; if we also insist that it is well-founded, we can apply the Mostowski Collapsing Lemma to get a transitive $\mathcal{V}'' = <V'', \in''>$ isomorphic to $<V', \in'>$. Since it is isomorphic to $\mathcal{V}'$, this model, too, satisfies exactly the same $\mathcal{L}_{\text{ZF}}$ sentences as does $\mathcal{V}$, and its membership relation, too, must be the real one (or behave just like it, which is the same thing). However, in $\mathcal{V}''$ all sets are transitive, and thus are either finite or countable, including the set representing the continuum. Thus, the continuum of this model is a *genuine* set, cut out of the intended model $\mathcal{V}$, and is *really* countable.

This stronger version of the paradox can be internalised. Let us remind ourselves briefly of how this proceeds.

Since the paradox essentially concerns modelling of the language $\mathcal{L}_{\text{ZF}}$, we have to have available some means of talking about the language of ZF inside ZF. This can be done via a standard coding which translates the relevant concepts into set-theoretical ones. Following this Gödelisation, it can then be shown that there is a one-place formula $F(x)$ of $\mathcal{L}_{\text{ZF}}$ which expresses the notion of being a formula of $\mathcal{L}_{\text{ZF}}$, i.e., holds of $x$ just in case $x$ is the code (set) of a formula of $\mathcal{L}_{\text{ZF}}$. This formula determines a set $F$, and we can show similarly that there are formulae and sets $Sen(x)$, $S$, $Ax(x)$, $A$ respectively, corresponding to sentences and axioms. With this as a basis, the standard model-theoretic elements can be reproduced fairly straightforwardly inside ZF. The most important of these is the three-place predicate $Sat(v_1, v_2, v_3)$, which says that $v_1$ satisfies $v_2$ inside $v_3$; this allows us to say, inside the theory, that a set $v_3$ is a model of a formula of $\mathcal{L}_{\text{ZF}}$, coded as $v_2$, under a satisfaction sequence $v_1$ (an eventually constant sequence of members of $v_3$). We can prove two central theorems inside ZF, first that $Sat$ obeys the Tarski conditions on satisfaction, and, following this, that:

$$\text{ZF} \vdash Val(\ulcorner \sigma \urcorner, u) \leftrightarrow \sigma^u \tag{10.1}$$

where "$Val(v_2, v_3)$" is the 2-place predicate obtained by universal quantification from $Sat(v_1, v_2, v_3)$, "$\sigma$" refers to a sentence of $\mathcal{L}_{\text{ZF}}$, "$\ulcorner \sigma \urcorner$" its code name, and "$\sigma^u$" refers to the standard relativisation of the sentence to the set $u$. What $Val(\ulcorner \sigma \urcorner, u)$ in effect says is that all satisfaction sequences drawn from $u$ satisfy $\sigma$ in $u$, and thus that "$\sigma$ is true in the interpretation $u$." In other words, $Val(\ulcorner \sigma \urcorner, u)$ is precisely the counterpart in ZF of the normal model-theoretic $<u, \in\restriction u> \models \sigma$. (10.1) shows the equivalence of this with the standard relativisation.

Using the notions expressed in "*Val*" and the predicates derived from it, it is now possible to express the notion of being a model of a set of sentences, to prove an internal version of the Completeness Theorem for first-order theories, and to express the notion of being an elementary substructure, written "$ES(x, y)$." With these in hand, we can now demonstrate the Downward Löwenheim-Skolem Theorem inside ZFC:

$$\text{ZFC} \vdash \forall y \subseteq u\big[[\aleph_0 \leq card(u)] \rightarrow \exists z[y \subseteq z \ \&\ card(z) = card(y) \ \&\ ES(z, u)]\big] \tag{DLST}$$

(Note the "ZFC"; this means that essential use is made of AC, just as with the conventional version.) On reflection, none of this is a great surprise, since one might argue that model theory is really nothing but informal set theory with a few rhetorical flourishes, and it would be utterly surprising if this conceptually straightforward piece of set-theoretical mathematics could *not* be rendered in our standard axiomatic set theories. But given (DLST), it is now not surprising that we can get some version of the Skolem Paradox, choosing $y$ such that $card(y) = \aleph_0$. But what we get now (under the *same* assumption that ZF is consistent) is neither contradictory nor paradoxical, but rather the connection with *absoluteness*, summed up in the following result concerning $C(x)$:

$$\text{If the system ZF is consistent, then the predicate } C(x) \text{ is not absolute.} \tag{10.2}$$

By the same token, $\neg C(x)$ fails to be absolute, too. Before explaining this in detail, we must first look at the notion of absoluteness.

Despite differences in the way absoluteness is presented, the notion, as the name suggests, is one of invariance or stability, the singling out of those terms or formulae which keep the *same* value (in the former case, set-value, in the latter, truth-value) as the domain in which they are evaluated varies.[13] Invariance itself, however, is not a fixed notion; more and more things will be absolute as more conditions are put on the domains to be used for the evaluation.

To illustrate this, the first definition of absoluteness (adapted from (Kunen, 1980, p. 117)) is as follows:

**Definition 1 (Absoluteness$_1$)** The formula $\psi$ with free variables $x_1, x_2, \ldots, x_n$ is said to be *absolute$_1$ for the formula* $\varphi$ if we can show:

$$\text{ZFC} \vdash \exists x\varphi(x) \ \&\ \forall x_1, x_2, \ldots, x_n\big[[\varphi(x_1) \ \&\ \varphi(x_2) \ \&\ \ldots \ \&\ \varphi(x_n)] \rightarrow [\psi^{\varphi}(x_1, x_2, \ldots, x_n) \leftrightarrow \psi(x_1, x_2, \ldots, x_n)]\big]$$

[13] In what follows, we will slip rather sloppily between talk of formulae, the domains they determine, and the extensions of the formulae, thus sets or proper classes.

This definition covers set terms as well: for the term $\tau$, just take the formula $x = \tau$. Here, $\varphi$ acts as the universe with respect to which the formula $\psi$ or the term $\tau$ is to be evaluated.

In this definition, nothing is said about the demands made by the formula $\varphi$, apart from there being something which satisfies it. Hence, it is unlikely that things satisfying a formula $\psi$ looked at from $\varphi$'s perspective will satisfy $\psi$ invariantly, i.e., be *absolute*$_1$ for $\varphi$. It is easy to show that absoluteness in this sense is preserved by the propositional connectives, but it is not difficult to exhibit very elementary notions which are not *absolute*$_1$; for example, the notion of something's being a subset of something else, i.e. "$x \subseteq y$." However, this formula *is* invariant in the sense we are investigating as soon as we demand that the evaluation take place in a *transitive* evaluative class $\varphi$; then the things which $\varphi$ "recognises" to be subsets are precisely the things which ZF can *prove* to be subsets. As soon as we demand transitivity, we can show that formulas built up from atomic formulas using only propositional connectives and *bounded* quantification, the so-called $\Delta_0$ formulas, are absolute as well.[14]

Another thing which may affect what is or is not absolute is the question of which principles of set theory are available in the class $\varphi$, that is to say, which principles $\varphi$ "recognises to be true." This is important because the definition of a term, say, and the proof that the set which the term attempts to define exists, will in general depend on the availability of some assumed background, thus on some of the axioms. Hence, these axioms ought to be available in the classes $\varphi$ if there is to be a fair assessment of invariance. For this, it is enough to look at models of the finitely many axioms used in the proof that the definition of the term in question is a good one, for the "background knowledge" required for a specific purpose is limited in exactly this way.

There is a perfectly good formal analogue to this informal talk of "availability." Saying that an axiom $\sigma$ is "available in $\varphi$" is really just to say that ZF $\vdash \sigma^{\varphi}$, for this latter, as (10.1) shows, really means that $\sigma$ holds in the structure $<\varphi, \in\restriction\varphi>$.[15] From this, there follows an obvious way in which the requirement of background knowledge can be inserted into the definition of absoluteness. Suppose the background knowledge which shows that a term is well-defined, or that a formula expresses what it is intended to express, is summed up in the axioms $\sigma_1, \ldots, \sigma_n$, then in assessing whether the formula or term is absolute for $\varphi$, we can demand that ZF $\vdash \sigma_1^{\varphi}$ & ... & $\sigma_n^{\varphi}$. In this way, we get the notion of an *absoluteness sequence* for a formula or term. Again, more things will become absolute if we do this.

Based on these observations, we can adopt a second, general definition of absoluteness:

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[14] See (Kunen, 1980, pp. 118–119).

[15] Whereas it is quite straightforward what $\sigma^{\varphi}$ means in ZF, speaking strictly "$<\varphi, \in\restriction\varphi>$" makes no sense, since the extension of $\varphi$ might not be a set. However, this abuse of notation is of a piece with the sloppiness pointed out in n. 13.

**Definition 2 (Absoluteness$_2$)** The formula $\psi$ (or term $\tau$) is said to be *absolute$_2$* if there is an absoluteness sequence $\sigma_1, \ldots, \sigma_n$ for $\psi$ (or for $x = \tau$) for which we can show, for each provably non-empty transitive formula $\varphi$, that if $ZF(C) \vdash \sigma_1^{\varphi}$ & ... & $\sigma_n^{\varphi}$, then $\psi$ is *absolute$_1$* with respect to $\varphi$.[16]

Note that we can show in ZF(C) the existence of set models of any finite collection of axioms $\sigma_1, \ldots, \sigma_n$, i.e., that ZF(C) $\vdash \exists x \left[\sigma_1^{\varphi} \,\&\, \ldots \,\&\, \sigma_n^{x}\right]$.[17] Hence, to show that a formula $\psi$ is not absolute$_2$, it suffices to show in ZFC that, for any sequence of axioms $\sigma_1, \ldots, \sigma_n$, there is a transitive set $u$ such that $\sigma_1^u$ & ... & $\sigma_n^u$, and such that the (i.e., ZFC's) evaluation of $\psi$ in $u$ differs from the evaluation of $\psi$ in the "universe" (i.e., ZFC's evaluation of $\psi$).

Note also that these are not the only notions of invariance which we might take as a sign that something is "absolute." For example, Gödel in his original work defined a formula $\varphi$ as absolute if its value in the constructible universe is the same as its actual value.[18] Nonetheless, *absoluteness$_2$* seems a fairly natural notion, perhaps the most natural which is not tied to a specific predicate like constructibility. The point here is that, even using such a weak notion of absoluteness, while $\omega$ is absolute, the theorem (10.2) asserts that $\neg C(x)$, and indeed $P(\omega)$ (ZF's term for the power set of $\omega$), are *not* absolute. The proofs which establish these failures of absoluteness proceed precisely by the reasoning which yields the Skolem Paradox. Let us outline the steps which lead to this.

Assume that ZF (and hence ZFC) is consistent. Now assume that $C(x)$ *is* absolute in the sense of *absoluteness$_2$*, and that $\sigma_1, \ldots, \sigma_n$ is an absoluteness sequence for it, a sequence we abbreviate by $\sigma$. ZF can prove Cantor's Theorem, from which it follows that $\exists x \neg C(x)$; let $\tau_1, \ldots, \tau_k$ be the list of axioms used in the proof of this, this time abbreviated by $\tau$. As was mentioned, we can always find set models of finitely many axioms. Hence, there must be an extensional set $z$ for which $\sigma^z$ and $\tau^z$ hold, thus showing that $z$ is a model for $\sigma$ and $\tau$. The set $z$ might be very large; but we can apply the (DLST) to cut it down to a countable, extensional subset $s$, an elementary substructure of $z$. Thus $\sigma^s$ and $\tau^s$ both hold, too. Moreover, since $s$ is extensional, the Mostowski Collapsing Lemma says that there must be a transitive set $u$ which is isomorphic to $s$; this is likewise countable, and it follows that $\sigma^u$ and $\tau^u$ also both hold. Now, since the axioms $\tau$ used in the proof of Cantor's Theorem hold in $z$, it is easy to see that $\exists x \neg C(x)$ holds relative to $z$, i.e., $[\exists x \neg C(x)]^z$, and the same must hold for the countable set $u$, for the axioms $\tau$ also hold there. Hence $[\exists x \neg C(x)]^u$, which in effect says that $u$ is a countable model of the statement "There exists an uncountable set," just as in the original Skolem Paradox. There is nothing contradictory about this in itself. But we can now use the absoluteness claim for $C(x)$ to *generate* a contradiction. $[\exists x \neg C(x)]^u$ is the same as $\exists x \in u \neg C^u(x)$; but since $u$ is transitive, and since also $\sigma^u$, the absoluteness of $C(x)$ entails that $\forall x \in u[C^u(x) \leftrightarrow C(x)]$. Hence, $\exists x \in u \neg C(x)$. On the other hand, since $x \in u$,

[16] This notion of absoluteness is taken from (Bell and Machover, 1977, p. 502).
[17] See (Kunen, 1980, p. 134ff.).
[18] See (Gödel, 1940, p. 42), or (Gödel, 1990, p. 76).

and $u$ is transitive, we must have $C(x)$, since $u$ is countable, from which it follows that there is an $x \in u$ for which $C(x)$ and $\neg C(x)$, which is a contradiction. This now gives the absurdity from which can conclude that $C(x)$ is not, after all, absolute. The argument that the term $P(\omega)$ is also non-absolute is entirely similar.

It is readily seen that this internal argument follows just the same pattern as the informal argument for Skolem's Paradox, except that, here, the assumption that $C(x)$ is absolute provides the genuine contradiction which the Skolem Paradox does not quite yield. Given this, the Skolem Paradox is transformed into a highly significant (meta-)theorem, parallel to the transformation of the genuine antinomies into theorems in the system.

As we have said, $\omega$ is absolute, and so is "being an ordinal number" (the predicate $Ord(x)$), but crucially not the predicate "being a cardinal number." What is the central difference?

The absoluteness of $\omega$ means that its value when evaluated in any $\varphi$ will be just $\omega$ itself, provided that $\varphi$ recognises the truth of the axioms in the absoluteness sequence for $\omega$. On the other hand, that $P(\omega)$ is not absolute means that, for any finite choice of axioms, the value of $P(\omega)$ will in general change as $\varphi$ varies; i.e., given any finite sequence of axioms, we can find a non-empty, transitive $\varphi$ satisfying these axioms such that $\varphi \cap P(\omega)$ is not the same as $P(\omega)$. Why does this happen?

The key is the presence of unbounded quantifiers, for these are what in general stand in the way of formulas being absolute.[19] As pointed out, a formula $\psi$ will be absolute for $\varphi$ (even in the sense of *absoluteness*$_1$) if $\varphi$ is transitive, and if $\psi$ contains no unbounded quantifiers. But whether or not the definition of a predicate $\psi$ or a set-term $t$ contains unbounded quantifiers depends crucially on the background knowledge available. To illustrate this, let us look at $\omega$.

The set $\omega$ is absolute, so the formula $x = \omega$ is an absolute formula. The usual proof of this proceeds by showing that the formula $x = \omega$ can be expressed using atomic formulae ($x = y$, $x \in y$), propositional connectives, and bounded quantifiers. To say that $x$ is $\omega$ is to say that $x$ is a limit ordinal while none of its members (predecessors) are, and all of this can be put in the right form providing the formula $Ord(x)$ contains only bounded quantification. But does it? On the face of it, the answer is negative. To say that $u$ is an ordinal is to say that $u$ is well-ordered by the $\in$-relation, and this involves an *unbounded* quantifier, for the special clause governing well-ordering begins by referring to all subsets of $u$, i.e., begins $\forall x\big[\forall w[w \in x \rightarrow w \in u \ldots$. The same holds if we want to define ordinals, not as certain well-ordered sets, but as certain *well-founded* sets, for the well-foundedness condition begins in exactly the same way. But now suppose the Axiom of Foundation is present; there is then no need to build in either the condition of being well-ordered by $\in$, or of being well-founded. It is enough to demand that $u$ be transitive and totally ordered by $\in$, and these notions can be expressed by formulae

[19] This seems to provide some link between the notions of absoluteness and impredicativity. See (George, 1987) for references to the idea that impredicativity concerns unbounded quantifiers. The link between the two notions is implicit in Poincaré's analyses of the antinomies given in §2. But as George points out, this cannot be all there is to the notion, which seems irredeemably imprecise.

which involve only *bounded* quantification, propositional connectives, and atomic formulae. Thus, it is clear that $x = \omega$ will be absolute for any $\varphi$ which is transitive and which satisfies the Axiom of Foundation, together with other bits of ZF (minus the power set axiom) which we require to prove that the definition is adequate.

What of $P(\omega)$? Can we "hide," in a similar way, any unbounded quantifiers involved in the specification of members of this? The answer is that we cannot.

Suppose $u$ is a transitive set which satisfies enough of ZF for it to be shown that $u$ contains a set $v$ which is the power set of $\omega$ as far as $u$ is concerned. It is important to note that the power set axiom taken on its own is quite weak, for it only says that all the subsets of a given set which the theory recognises can be collected together into a set, and this is quite consistent with its being provable that only very few subsets exist. However, the power set axiom gets its strength by being combined with the Axiom of Separation (Zermelo's *Aussonderungsaxiom*), which (in first-order set theory) is the schema:

$$\forall x \exists y \forall z [z \in y \leftrightarrow (z \in x \ \& \ \psi(z))] \qquad \text{(SEP)}$$

This holds for any $\psi$ in the language, regardless of how formed, thus, in particular, for any $\psi$ containing unbounded quantifiers.[20] Thus, $v$ will contain only those sets which the "model" $u$ can recognise as subsets of $\omega$, and it is clear that this need not be all the genuine subsets of $\omega$. Suppose that $y \subseteq \omega$, and that $y$ is given by an application of separation to a formula $\psi(z)$ which contains unbounded quantifiers. We can think of $\psi$ and the condition "$z \in \omega$" coalesced into the one condition $\varphi$, in other words, that $y$ is given by the abstraction term $\{z : \varphi(z)\}$. Allow that $u$ is a model of those axioms needed to prove that $\{z : \varphi(z)\}$ defines a set, in particular the requisite instance of (SEP). The key question now is whether $v$ has $y$ as a member, or, in other words, how $u$ will evaluate the abstraction term $\{z : \varphi(z)\}$. It is immediately obvious that $u$ will evaluate this term as $\{z \in u : \varphi^u(z)\}$, and hence that all the unbounded quantifiers in $\varphi$ will now be bounded by $u$. Thus, we now have quite a different abstraction term from $\{z : \varphi(z)\}$, one which *a priori* we have no reason to think corresponds to $y$ itself, although it yields a subset of $y$. Thus, although $v$ will certainly contain $y^u = \{z \in u : \varphi^u(z)\}$, it is quite possible that it will fail to contain $y = \{z : \varphi(z)\}$, hence, quite possibly will not contain members matching all the subsets of $\omega$.

What this suggests is that $u$ will not "recognise" the existence of a set (or rather, the full extension of the set) corresponding to an abstraction term which contains unbounded quantifiers, for it will automatically read those quantifiers as being bound by $u$. However, (SEP) tells us that there are subsets of $\omega$ given by abstraction terms which have quantifiers which range beyond $u$. Thus, apparently, any attempt to pin down the extent of $P(\omega)$ which only takes into account formulas $\psi$ restricted

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20 Let us note in passing that, in general, axiom systems are not the simple sum of their axioms, but that the axioms (so to speak) co-operate. Thus, although apparently weak on its own, the power-set axiom is enormously powerful when combined with the Axioms of Separation (or Replacement) and Infinity.

to what is inside $u$, and not the full range of the $\psi$ which (SEP) permits, will not succeed. In this sense, then, the extent of $P(\omega)$ does seem to depend on that of the "richness of the universe."[21]

To summarise, it seems to me that this "internalisation" both disperses the mystery which is sometimes glimpsed as a result of the Skolem Paradox, and actually to a large extent explains it. It converts the paradox into a genuine contradiction, which is then exploited via an appropriate *reductio*. Moreover, in doing so, Poincaré's existence principles (LST and IST) are firmly rejected, as is clear in the formulation of (SEP); consequently, many sets/extensions which were taken as illegitimate by Poincaré are taken to exist as sets in an unproblematic way with perfectly determinate extensions. Given this, Poincaré's worry about variation in the extension of sets translates directly into the property of absoluteness. In particular, this argument shows that there is a radical difference between the set of natural numbers and the central focus of Poincaré's disquiet, the Cantor-Zermelo continuum, the former being absolute, and the latter not, making it clear that there is a sharp separation between the determinateness of the continuum and its non-absoluteness. This is what, in the end, Zermelo's insistence on the cogency of impredicative definitions amounts to.

These results, of course, are far from paradoxical. But although the internalisation of the Skolem Paradox both dismisses Poincaré's worries about incoherence and takes away any sense of paradox, it does serve to shift attention to a genuine conceptual difficulty within the theory of sets, namely that the cardinal notion of uncountability is insufficiently tied to the ordinal notions which Cantor had adopted to explain it. This is revealed by both Gödel's proof of the consistency of the Generalised Continuum Hypothesis (GCH) and Cohen's proof of the independence of the Continuum Hypothesis (CH). In these consistency and independence proofs, the notion of non-absoluteness (and indeed something close to the Skolem Paradox) plays a central role. A quick inspection of the main lines of argument will show this. It is to this that I turn in the final section.

## 4 Absoluteness, Consistency and Independence

Let us look first at Gödel's consistency proof, for which the technical notion of absoluteness was first introduced.

The key to the proof is the constructibility predicate $L(x)$ and the notion of constructible subset on which it is based. The constructible subsets of a given set $u$ are, in effect, the subsets of $u$ definable in $u$, in other words, just the subsets of $u$ which satisfy a defining formula whose quantifiers are restricted to $u$. This loose account can be rendered inside $\mathcal{L}_{\mathrm{ZF}}$ itself, given the formal notion of satisfaction. Let $Sat(x, y, z)$ be the satisfaction predicate described before, and let $ec(u)$ be the set of eventually constant sequences of $u$, effectively the set of satisfaction sequences

[21] This way of putting the matter is taken from (Bell and Machover, 1977, p. 509).

defined over $u$. (Note that, if $u$ is infinite, $ec(u)$ has the same cardinality as $u$.) We can then say that a subset $v$ of $u$ is a constructible subset of $u$ if there is a $z \in F$ and a $w \in ec(u)$ such that

$$v = \{y : y \in u \ \& \ Sat(w(0/y), z, u)\}$$

where $w(0/y)$ denotes the sequence $w$ with the only change (if any) being that $w$'s 0-th choice is replaced by $y$. Thus, $v$ is the collection of all those $y$ for which there is a formula $z$ and a satisfaction sequence whose first element is $y$ satisfying the selected formula inside $u$.

The crucial thing about this is that satisfaction has to take place inside $u$, in other words, given one of the central adequacy theorems concerning the predicate $Sat$, a definable subset $v$ of $u$ is just the set of all $y$ in $u$ which satisfy some formula $\varphi$ whose quantifiers are restricted to $u$. But this just means, in effect, that $v = \{y \in u : \varphi^u(y)\}$, and that $v$ is a collection of members of $u$ picked out in a way that does not make reference, either explicitly or implicitly, to sets which are not already present in $u$ itself. The abstraction term for $v$ is just what the abstraction term $\{y : \varphi(y)\}$ becomes when it is evaluated in $u$.

These subsets of $u$ can now be collected together into a set, $D(u)$, often called *the predicative power set of* $u$, meaning that it consists of just those subsets of $u$ which are capable of definition using only quantification restricted to $u$ itself. In usual predicative systems, such as that of Russell, quantification is restricted to some stage or type, and the object being defined is then taken to belong to the next stage or type. Here $u$ should be regarded as the stage or type itself, which is what we get, in effect, if we demand that types be cumulative.

Not unexpectedly, what we now find is that, quite unlike the full power set $P(u)$, if the set $u$ is infinite, then the predicative power set $D(u)$ has the same cardinality as the starting set $u$, the number of formulas available for use in definitions (thus, the size of $F$) being only countably infinite, and both $ec(u)$ and the collection of parameters available have the same size as $u$. Moreover, whereas $P(u)$ is not absolute, for infinite $u$, $D(u)$ is, which is no surprise, given the bounded nature of the quantification. Hence, for any non-empty transitive $\varphi$ which satisfies the absoluteness sequence for $D(u)$, $\varphi \cap D(u)$ is equal to $D(u)$.

The predicate $L(x)$ is now defined with the help of the operation $D(x)$. First, sets $L_\alpha$ are defined for all ordinals using a transfinite recursion; for successor ordinals $\alpha = \beta + 1$, $L_\alpha = D(L_\beta)$, and when $\alpha$ is a limit ordinal $L_\alpha = \bigcup\{L_\beta : \beta < \alpha\}$. Hence, the $L_\alpha$ form a cumulative hierarchy, the so-called *constructible hierarchy*. (In sum, for all ordinals $\alpha$, $L_\alpha = \bigcup\{D(L_\beta) : \beta < \alpha\}$.) Since the operations that go into this recursion (in particular, the $D(x)$ construction and the predicate $Ord(x)$) are all absolute, we can show that both $x = L_\alpha$ and $x \in L_\alpha$ are absolute formulas too. The definition of $L(x)$ is now simply:

$$L(x) \leftrightarrow \exists\alpha[Ord(\alpha) \ \& \ x \in L_\alpha]$$

The formula for $L(x)$ contains an unbounded quantifier, so is not absolute in our sense, although it is invariant for transitive models of ZF which contain all the ordinals. Nevertheless, the absoluteness of $x = L_\alpha$ and $x \in L_\alpha$ is enough to be able to prove the central relative consistency results, as we will see.

It can now be shown that the constructible sets form an "inner model" of set theory in which both the Axiom of Choice (AC) and the GCH hold. To show this requires showing three things[22]:

(1) Each axiom $\sigma$ of ZF (AC is *not* included) is provable in ZF when relativised to the predicate $L$, i.e., ZF $\vdash \sigma^L$.
(2) The statement that all sets are constructible, i.e., $\forall x \exists \alpha [x \in L_\alpha]$ (always abbreviated as V = L) is also provable in ZF when relativised to the predicate $L$, thus ZF $\vdash$ (V = L)$^L$.

(1) and (2) together are enough to show that V = L is consistent relative to ZF. It is now shown that inside ZF both AC and GCH follow from V = L, i.e.,

(3) ZF + V = L $\vdash$ AC and also ZF + V = L $\vdash$ GCH.

Hence, AC and GCH must also be consistent relative to ZF.

What interests us here is (3), or rather that part of it which concerns the derivation of GCH from V = L, for this very closely mirrors the argument behind the internalised Skolem Paradox. Key use is made of the fact that the formula $x \in L_\alpha$ *is* absolute, which turns on the fact that predicative power-set formation, unlike full power-set formation, is absolute. The heart of the proof is the Lemma, which Gödel calls "an axiom of reducibility for sufficiently high orders" (Gödel, 1938, p. 556), that if all sets are constructible (i.e., if V = L), then any subset of $L_{\aleph_\alpha}$ (i.e., any member of $P(L_{\aleph_\alpha})$) is not just *somewhere* in the $L$-hierarchy, but must be constructible *before* the stage $L_{\aleph_{\alpha+1}}$. This time we take an absoluteness sequence $\sigma$ for $x \in L_\beta$, in much the same way as before we started with a supposed absoluteness sequence for $C(x)$, and we assume V = L is in $\sigma$. Take any subset $u$ of $L_{\aleph_\alpha}$, and consider the set $L_{\aleph_\alpha} \cup \{u\}$. There must be a transitive set $v$ which includes $L_{\aleph_\alpha} \cup \{u\}$ for which $\sigma^v$ & [V = L]$^v$. Since it is easily proved that $card(L_{\aleph_\alpha}) = \aleph_\alpha$, then $card(L_{\aleph_\alpha} \cup \{u\})$ is $\aleph_\alpha$, too. Hence, applying the (DLST), just as before, we can find an elementary substructure $w$ of $v$ which is of size $\aleph_\alpha$ and which also includes $L_{\aleph_\alpha} \cup \{u\}$. (The only difference here is that we have focused on $\aleph_\alpha$ rather than $\aleph_0$.) Hence, $\sigma^w$ & [V = L]$^w$. It can be argued easily that, although $w$ need not be transitive, it must be extensional since $v$ is transitive, and thus must satisfy the Axiom of Extensionality. Therefore $w$ also satisfies it, so we can apply the Mostowski Collapsing Lemma to collapse $w$ to a transitive set $x$ isomorphic to it. But since $L_{\aleph_\alpha} \subseteq w$ and $L_{\aleph_\alpha}$ is transitive, the collapsing isomorphism applied to each member of $L_{\aleph_\alpha}$ (thus each member of $u$) must be the identity. Hence, it follows that $u$ itself must be a subset of $x$.

[22] See (Bell and Machover, 1977, pp. 480–481).

Since now $[\mathrm{V}=\mathrm{L}]^x$, this must amount to $\forall y \in x \exists \beta \in x[y \in L_\beta]^x$ (appealing to various facts about transitive sets). This is where the absoluteness of the formula $y \in L_\beta$ comes in, for this allows us to replace "$[y \in L_\beta]^x$" with "$y \in L_\beta$" in the formula, yielding $\forall y \in x \exists \beta \in x[y \in L_\beta]$. Since $u \in x$, then $\exists \beta \in x[u \in L_\beta]$. Let $\beta \in x$ with $u \in L_\beta$. But since $x$ is transitive, this must mean that $\beta \subseteq x$. Hence, $card(\beta) \leq card(x) = \aleph_\alpha$, and so $\beta < \aleph_{\alpha+1}$, which means that $L_\beta \subseteq L_{\aleph_{\alpha+1}}$. Hence, $u$, the subset of $L_{\aleph_\alpha}$ we are considering, must be a member of $L_{\aleph_{\alpha+1}}$, and hence constructible before the stage $L_{\aleph_{\alpha+1}}$. Accordingly, we have proved that

$$\mathrm{ZF} + \mathrm{V} = \mathrm{L} \vdash P(L_{\aleph_\alpha}) \subseteq L_{\aleph_{\alpha+1}},$$

which means that

$$\mathrm{ZF} + \mathrm{V} = \mathrm{L} \vdash 2^{\aleph_\alpha} \leq \aleph_{\alpha+1};$$

and, since it is an elementary theorem of ZFC that $\aleph_{\alpha+1} \leq 2^{\aleph_\alpha}$, it follows that

$$\mathrm{ZF} + \mathrm{V} = \mathrm{L} \vdash 2^{\aleph_\alpha} = \aleph_{\alpha+1};$$

thus the GCH.[23]

Hence, we might say that the availability of the machinery which enables us to "internalise" the Skolem argument is *precisely* that which is deployed in the result central to showing the relative consistency of the GCH. And the non-absoluteness of $C(x)$ (or of the power-set operation), or alternatively, the absoluteness of the predicative power-set operation, is central to the proof.[24]

Skolem's original 1923 paper shows remarkable prescience in matters concerning the consistency and independence phenomena. According to Skolem, Zermelo's axiomatisation rests on a logically prior notion of "domain",[25] and Skolem clearly thinks that there is something odd, if not circular, in trying to found the notion of set

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[23] For clear accounts of the proof of AC from ZF + V = L, as well as Gödel's main "reducibility lemma," see (Kunen, 1980, pp. 174–175) or (Bell and Machover, 1977, pp. 517–522). Bell and Machover point out the similarity of the proof of the main lemma with the proof of the non-absoluteness of $P(x)$; see p. 522.

[24] It is worth pointing out that it is the iteration of $D(x)$ through the classical ordinals that prevents the constructible hierarchy of the $L_\alpha$ being an appropriate setting for predicative mathematics. See for example (Kreisel, 1960, p. 386). According to what Kreisel calls "the fundamental idea of predicativity" (ibid., p. 387), an ordinal $\alpha$ is predicatively legitimate for use in defining a given level $\Sigma_\alpha$ of predicative definitions if there is a lower level $\Sigma_\beta$ (with $\beta < \alpha$) such that there is a well-ordering of the natural numbers of type $\alpha$ definable by a formula from $\Sigma_\beta$. (See also (Gödel, 1944; Wang, 1954).) This is not in general true of the $L$-hierarchy.

[25] In his first paper on the axiomatisation of set theory (Zermelo, 1908b, p. 262), Zermelo says the following:

> Set theory is concerned with a "*domain*" $\mathfrak{B}$ of objects, which we will call "*things*", and among which are the sets.

on something set-like, and fully unexplained, like the notion of domain. In particular, he takes it that this rules out the use of the set-theoretic axioms in investigating dependence phenomena for the set-theoretic system itself, useful and fruitful as this might be for investigating these questions for logical systems in general. This is because an assumption of consistency would have to be made, and this would in turn rest on the assumption that "domains" exist, which, properly speaking, would demand axioms for the notion of domain, potentially leading to an infinite regress. Thus:

> If one is not to base oneself again on axioms for domains (and so on, ad infinitum), I see no other choice but to turn to considerations like those which were applied above in the proof of Löwenheim's Theorem, for which the idea [*Vorstellung*] of the finite whole number is assumed as the basis.[26]

Gödel's result has a rather different conceptual basis, in particular because it is consciously a *relative* consistency proof, and one pursued by the method of inner models first used for set theory by von Neumann in (von Neumann, 1929). But crucially it has at its core precisely an application of "Löwenheim's Theorem", as Gödel himself pointed out:

> This ['reducibility'] lemma is proved by a generalization of Skolem's method for constructing enumerable models. (Gödel, 1939, p. 93.)

Skolem's comments are prescient, not just with respect to Gödel's work, but also with respect to Cohen's later proof of the independence of CH. Skolem considers the question of whether the Zermelo axioms isolate a unique domain (up to isomorphism), for clearly the conceptual dependence of set theory on a notion of domain would then be somewhat less serious. He suggests how it might be shown that this is not the case, namely by taking a domain $B$, and trying to adjoin to it a set $a \notin B$ in something like the way a new object is adjoined to an algebraic structure, though the set theory case will be a good deal more involved. Skolem then says the following:

> Much more interesting would be to be able to prove that one can adjoin a new subset of $Z_0$ [Zermelo's set of natural numbers] without giving rise to contradictions. This however will be very difficult.[27]

In a footnote, Skolem then goes on:

> Since the Zermelo axioms do not determine the domain $B$ [of all sets] uniquely, it is very unlikely that all problems concerning powers will be decidable using these axioms. For example, it is quite possible that the so-called Continuum Problem, namely whether $2^{\aleph_0} >$ or $= \aleph_1$ is simply not solvable on this basis; nothing need be decided about it. It could be that the situation here is just the same as in the following case: an undetermined field [*Rationalitätsbereich*] is given, and one asks whether there is present in this domain a

[26] Skolem (1923, pp. 229–230) (English translation, p. 299). See also (Wang, 1970, p. 39).
[27] Skolem (1923, p. 229) (English translation, pp. 298–299).

> magnitude [*Grösse*] $x$ such that $x^2 = 2$. Because of the ambiguity of the domain this is just not determined.[28]

None of this can be called an anticipation of the Cohen *methods*, but the idea of adjoining a set is a fair general description of Cohen's *goal*, perhaps most easily seen in the adjunction of a non-constructible set to a model which satisfies V = L. The genius of Cohen's work is showing how it is possible to adjoin sets to a model of ZF in such a way that one obtains a new model of ZF. In doing this, the construction of both countably infinite interpretations and of absoluteness again play key roles. In the following pages, I will try to say what this connection is, by giving a very brief sketch of Cohen's proof.

Assume we have a countable, transitive model $<M, E>$ of ZF, the basis of the Skolem argument in the sharpened form. Since $<M, E>$ is countable and transitive, its continuum $c^M$ must be countable. But the cardinal number $\aleph_2$ of this model must also be countable; hence, it seems that there must be a function $f$ which is a bijection between $c^M$ and $\aleph_2^M$ (or $\omega_2^M$), even though $M$ cannot recognise this function. Suppose we can now produce a new structure $<N, E'>$ which preserves as much of $<M, E>$ as possible, enough anyway to remain a model of ZF, but which also *contains* $f$ as an element. Then $<N, E'>$ will be a model of ZF together with the sentence 'The cardinality of the continuum is $\aleph_2$,' that is, providing $<N, E'>$ is constructed properly. This (very crudely put) is the basis of the Cohen independence proofs. Note that the very starting point for the idea of Cohen's proof is rooted in the phenomenon which lies at the basis of the Skolem Paradox, namely the ((DLST)) and the existence of countable models.[29]

Let us try to be a bit less crude. First of all, we can use ZFC itself to carry out the construction, since $<M, E>$ does not have to be a model of the whole system, but only of a finite fragment of it, namely of the axioms which we actually use. If we list these axioms, then ZF can prove that there is a model based on a set $M$ which satisfies them, meaning in particular that we can regard the relation $E$ as just $\in$. Just as before, we can use the (DLST) and the Mostowski Collapsing Lemma in tandem to produce a countable, transitive set which also satisfies these axioms. Thus, in effect, we can regard $M$ as a countable, transitive model of enough of ZF as we need. What the Cohen technique does is to show how to take such an $M$ and create

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[28] Skolem (1923, p. 229) (299 of the English translation). Skolem's original has "Alef" and not "$\aleph$." van der Waerden (1937, p. 40) gives "Rationalitätsbereich" as an alternative to "*Körper* [field]." The English translation of Skolem's paper has "commutative field."

[29] As Cohen himself said (Cohen, 2005, p. 2417):

> For example, he [Skolem] pointed out the existence of countable models of set theory. ... But certainly he was aware of the limitations on what can be proved. In a remarkable passage, he even discuses how new models of set theory might be constructed by adding sets having special properties, although he says he has no idea how this might be done. This was exactly the starting point for my own work on set theory, although I was totally unaware that Skolem had considered the same possibility.

For a more detailed discussion, see (Kanamori, 2008).

in ZF an expansion $N$, still countable and still a model of ZF (or the bits that we need), which contains the same ordinals as $M$, but also contains a function like $f$ showing that the continuum in the expansion is equinumerous with its second aleph, thus violating the CH.

Suppose that $p$ is an arbitrary partially ordered set in the countable, transitive model $M$. Let $G \subseteq p$ be a filter in $p$; $G$ is said to be *generic* if its intersection with any set *dense* for $p$ in $M$ is non-empty. ($d$ is said to be dense for $p$ in $M$ if $d \in M$, and if $\forall x \in p \exists y \in d[y \leqslant x]$.) Cohen showed that for each $p$ there will always be a generic set. Since $M$ is transitive, we have $G \subseteq M$, but in general $G$ will not belong to $M$ as $p$ does. If it does not, then we can define the smallest model $N$ (often denoted by $M[G]$) which contains $M$ and which also has $G$ as a member. ($G$ will be the new subset adjoined to $<M, E>$, or the basis of the new function, which amounts to the same.) Cohen showed that this $M[G]$ exists for generic $G$, and indeed is also a countable, transitive model of ZF if $M$ is, and indeed contains exactly the same ordinals. The existence of $M[G]$ is exactly what the forcing construction shows; the members of $p$ are at the basis of this, the so-called forcing conditions. Note that since $G$ is in $M[G]$, and $M[G]$ is a model of ZF, then $[\bigcup G]^{M[G]}$ is in $M[G]$. But union is an absolute operation, so $[\bigcup G]^{M[G]} = \bigcup G$, and this set is therefore in $M[G]$.

So far, all of this is quite general. The particular details of what $M[G]$ will satisfy over and above the axioms of ZF will depend on exactly what are chosen as the forcing conditions. Let us go back to our bijection $f$ between $\omega_2^M$ and $c^M$. So far as we know, there is no such $f$ in $M$; indeed, we can insist that $M$ satisfies the GCH so that we know there cannot be. We can put $[P(\omega)]^M$ instead of $c^M$, because these two sets have the same cardinality in $M$, as does the set of all functions (characteristic functions) from $\omega^M$ to $2^M$, i.e., the set $[2^M]^{\omega^M}$. There will of course be plenty of maps $g$ in $M$ going between $\omega_2^M \times \omega^M$ and $2^M$; but since $M$ possesses no injection between $\omega_2^M$ and $[P(\omega)]^M$, then none of these maps will be such that the corresponding $g^* : \omega_2^M \to [2^M]^{\omega^M}$ is injective. However, the finite fragments of such a map will be in $M$, and indeed so will be the set of all these finite fragments. Thus put

$$\mathrm{Fn}\left(\omega_2^M \times \omega^M, 2^M\right) = \left\{p : card(p) < \omega^M \wedge \mathrm{dom}(p) \subseteq \omega_2^M \times \omega^M \wedge \mathrm{ran}(p) \subseteq 2^M\right\}$$

This Fn $\in M$. So, in a sense, $M$ recognises all the finite approximations to a function $f$ like the one we want to consider, even though it cannot recognise such a function itself.

Fn is partially ordered by reverse inclusion, i.e., $x \leq y$ iff $x \supseteq y$, so we can take Fn as the set of forcing conditions. Now suppose $G$ is any filter on Fn. Then $\bigcup G$ must be a function from $\mathrm{dom}(\bigcup G) \subseteq \omega_2^M \times \omega^M$ and $\mathrm{ran}(\bigcup G) \subseteq 2^M$. But the sets

$$D_i = \left\{ p \in \mathrm{Fn}\left(\omega_2^M \times \omega^M, 2^M\right) : i \in \mathrm{dom}(p) \right\}$$
$$D_j = \left\{ p \in \mathrm{Fn}\left(\omega_2^M \times \omega^M, 2^M\right) : j \in \mathrm{ran}(p) \right\}$$

are all dense, which must mean that if $G$ is generic, it intersects them all, and we can conclude from this straightforwardly that for a generic $G$, $\mathrm{dom}(\bigcup G) = \omega_2^M \times \omega^M$ and $\mathrm{ran}(\bigcup G) = 2^M$, which means that $\bigcup G$ is a function from $\omega_2^M \times \omega^M$ onto $2^M$.

We can now easily show that this $\bigcup G$ generates a function $f$ of the kind we are seeking. For each $\alpha$ in $\omega_2^M$, we get a function $f_\alpha$ from $\omega^M$ to $2^M$ (given by $f_\alpha(n^M) = \bigcup G(\alpha, n^M)$), and we can show that if $\alpha \neq \beta$, then $f_\alpha \neq f_\beta$. In other words, as the $\alpha$ run through $\omega_2^M$, the $f_\alpha$ run through distinct subsets of $\omega^M$. $\bigcup G$ ($= f$) is clearly just the amalgamation of the $f_\alpha$. Hence, in $M[G]$ we can show that there are $\omega_2^M$ *different* subsets of $\omega^M$.

It is at this point that we can see the importance of the absoluteness of $\omega$ and the non-absoluteness of $P(\omega)$. We know that $\omega$, like 2 (and any other finite ordinal), is absolute, so therefore $n^M$ and $\omega^M$ are the same in $M[G]$, i.e., just $n$ and $\omega$ respectively. Thus, $M[G]$ knows that there are $\omega_2^M$ subsets of $\omega$. But this will only give us a violation of CH if we know that $\omega_2^{M[G]}$ is the same as $\omega_2^M$, in other words, if we know that the shift from $M$ to $M[G]$ preserves this particular uncountable cardinal. We know that cardinality, as opposed to ordinality, is not an absolute notion, although it turns out that in this particular construction $\omega_2^M$ *does* equal $\omega_2^{M[G]}$. Thus CH is violated in $M[G]$; $M[G]$ adds subsets of $\omega$ to $M$'s $[P(\omega)]^M$.

We can now see the importance of the fact that the continuum is not absolute, for the cardinality of $M[G]$'s continuum is $\aleph_2$ (in $M[G]$), and we have seen that this cardinal is the same in both $M$ and $M[G]$. But the cardinality of the continuum in $M$ is *not* $\aleph_2$ because $M$ satisfies GCH. Hence it must be the case that $c^M \neq c^{M[G]}$, which could not be the case if $P(\omega)$ *were* absolute. Thus, we have traded essentially on the fact $\omega$ is absolute but $P(\omega)$ is not.

The method roughly sketched here is extremely flexible. For one thing, by varying the finite partial functions chosen, one can produce models $M[G]$ in which the power of the continuum is almost any uncountable cardinal. For another, even the stability of the uncountable cardinal chosen is something which depends on the forcing conditions. That $\omega_2^M$ equals $\omega_2^{M[G]}$ is a consequence of the fact that Fn satisfies the *countable chain condition* (c. c. c.) in $M$.[30] In fact, if $M[G]$ preserves cofinalities (i.e., if $\mathrm{cf}(\gamma)^M = \mathrm{cf}(\gamma)^{M[G]}$), then any cardinal in $M$ will also be a cardinal in $M[G]$, and if the set of forcing conditions $p$ satisfies c. c. c. in $M$, then

[30] A *chain* in a partially ordered set $p$ is a subset of $p$ in which the ordering relation is total. Two elements $x, y$ of $p$ are *compatible* if there is an element $z$ of $p$ such that $z \leq x, z \leq y$, and incompatible if there is no such $z$. An *antichain* in $p$ is a subset $q$ of $p$ such that any two distinct elements of $q$ are incompatible. The c. c. c. for $p$ then says that any antichain of $p$ is countable. See (Kunen, 1980, p. 53).

cofinalities *will* be preserved in the shift to $M[G]$.[31] This is important to realise, since by choosing sets of forcing conditions which do not have c. c. c., one can violate cardinal preservation, or indeed 'collapse' cardinals in $M$ to smaller ordinals in $M[G]$. In short, the forcing models show how radically the size of the continuum is undetermined by the standard axioms of ZF, thereby underlining the "relativity" of the central notions of cardinality and power.

## 5 Conclusion

The central focus of this paper is the internalisation of the Skolem Paradox, showing how this latter is transformed from a puzzle into an important technical result, a result which, I have argued, is fundamental in the study of modern axiomatic set theory. I have also suggested that the result is a clear reflection, in a very different setting, of what is at the heart of Poincaré's diagnosis of the antinomies, namely that the Cantorian continuum possesses an instability property not possessed by the collection of natural numbers. Poincaré wished to demonstrate, through his analysis, that the Cantorian continuum, indeed "Cantorism" generally, is incoherent. Modern set theory rejects that conclusion. In fact, it might be argued that the non-absoluteness of $P(\omega)$ shows that set theory gives a sufficiently refined account of the continuum to recognise a sharp conceptual distinction between the continuum and the set of natural numbers. Nevertheless, the internalisation of the Skolem Paradox ultimately shows that Poincaré's diagnosis, when reflected in the way I have indicated, points to a central conceptual difficulty, namely set theory's inability to solve the most basic non-trivial questions about exponentiation in the ordinal theory of infinite power.[32]

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[31] See (Kunen, 1980, pp. 204–208).

[32] The material in this paper has been through many incarnations over the last 20 years. The inspiration for it, though, is John Bell, who first made me aware of the internalised version of the Skolem Paradox. For discussions on this and related matters, I am also deeply indebted to the late George Boolos, William Demopoulos, Michael Friedman, Moshé Machover, the late John Macnamara, Mihaly Makkai, Stephen Menn and Gonzalo Reyes. As acknowledged in n. 3, I also owe a substantial intellectual debt to Wilfrid Hodges. I am also very grateful for the generous support of the Social Sciences and Humanities Research Council of Canada over many years, as well as the FQRSC of Québec, formerly FCAR. It is also a pleasure to acknowledge the gracious support of the Alexander von Humboldt Stiftung and the Akademie der Wissenschaften zu Göttingen. Note that the translations which appear in the text are my own, even where published translations are referred to as well as the originals.

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