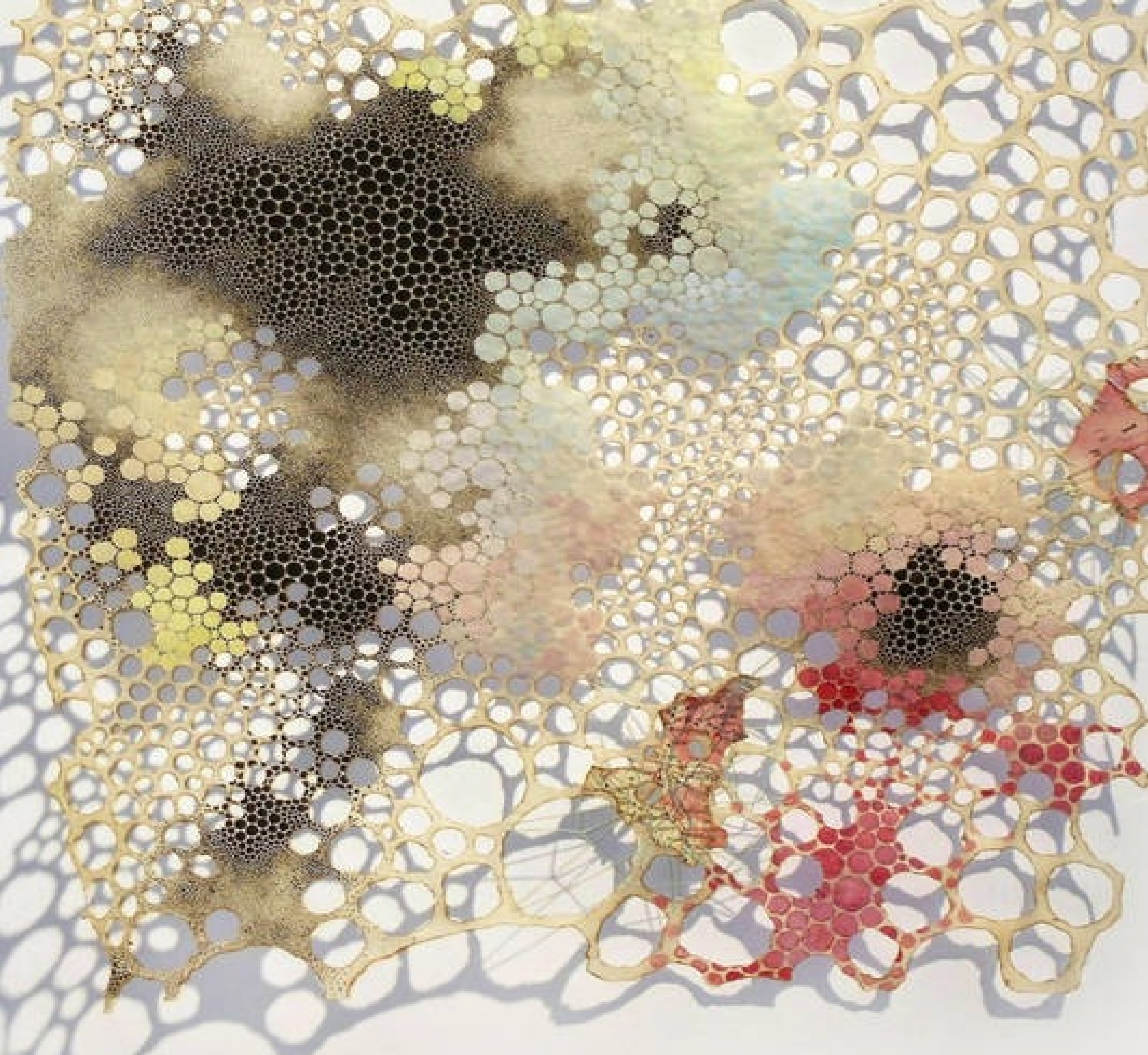


CULTURES WITHOUT CULTURALISM

The Making of Scientific Knowledge

KARINE CHEMLA and
EVELYN FOX KELLER, editors

KARINE CHEMLA

14

Changing Mathematical Cultures, Conceptual History, and the Circulation of Knowledge

A Case Study Based on Mathematical Sources from Ancient China

Why should we attend to the various scientific cultures to which the different sources that provide evidence of scientific activity attest? The ways in which given social groups practice, and have practiced, science do not fall out of the sky—collectives of practitioners shaped them and shared them. Collectively they made them change, especially in relation to questions they were considering. In this sense, these "ways of practicing science" are an outcome of scientific activity, along with concepts, results, and theories. In the first approximation, it is to these ways of practicing science that I refer when using the term "cultures." One of the obvious reasons they should matter to historians is that they are a result of scientific activity. This is not, however, my main topic here.

In this chapter, I concentrate on another important reason that we should take scientific—or, more generally, scholarly—cultures into consideration. The thesis I propose is that the description of such cultures is an essential endeavor inasmuch as it provides historians with tools to interpret, in a more rigorous way, writings produced in the framework of these cultures. Cultures, in this sense, grasp a kind of context for scientific activity that gets quite close to actors' practice. What do I mean by this? Understanding various aspects of scientific activity requires that we take into account distinct types of context, on different scales. My focus here is on a micro-level. I aim to concentrate on the detailed actions of actors, thus examining these collectives from a specific angle. In the ancient time period I consider, shared ways of doing science to which our sources attest are often the only means available to perceive collectives within which the documents were produced.

The difficulties attached to interpretation become striking when one deals with Chinese, Sanskrit, or Mesopotamian sources produced millennia ago. This does not mean these problems arise only for such documents. However, dealing with these sources magnifies some of the theoretical challenges posed by interpretation, thus requiring that these issues be addressed. Indeed, I am convinced that the lack of attention paid to these issues has contributed to the relative disparagement of these sources in the history of science. I argue here that a focus on mathematical cultures, through the close context it allows us to capture, helps us solve problems of interpretation, giving us clues for grasping concepts and perceiving results in our sources. What I am suggesting here is that for the history of science the description of cultures provides a tool for the practice of conceptual history.

I illustrate these claims on the basis of a single example. I concentrate on a corpus of Chinese sources, ranging from the first to the thirteenth century, that bears witness to a given tradition. Yet I argue that these sources attest to different mathematical cultures, despite significant overlap among them. I show that the description of the mathematical culture to which the earliest documents adhere allows us to identify in them a concept of quadratic equation, in a sense I shall make clear. I further argue that one can likewise perceive in that tradition, at different time periods, different yet related concepts for what we call quadratic equations.[1] We shall see that the description of mathematical cultures allows us to interpret the sources, and also to attend to the concepts to which Chinese sources attest, as well as to the ways in which practitioners worked with them.

The case study allows me to argue for several other theses regarding scholarly cultures. First, the fact that several mathematical cultures can be identified in the tradition examined between the first and the thirteenth centuries show that scholarly cultures change over time, displaying continuities with previous practices as well as breaks. The breaks partly reflect general changes in the larger environment in which mathematical practice is carried out. They partly echo actors' collective transformation of their way of working. Talking in terms of cultures thus does not imply that we deal with unchanging, static entities. I argue that historians need to take account of these changes in their practices of interpretation.

Second, this reflection on interpretation highlights another issue: my approach to this part of the history of algebraic equations shows that the concepts of equation identified at different time periods all have specificities that can be correlated with features of the mathematical culture in relation to which they were shaped. Note that this explains why the description of cultures provides tools for interpretation. There is no determinism here. We shall see that practices change, that the treatment and understanding of equations undergo transformation, and that the correlation between the two is not a deterministic one. Instead, I suggest that the correlation indicates that cultures also change partly in relation to the conceptual work done on equations as much as the concepts change in relation to how actors worked. Cultures thus prove useful not only as a tool of interpretation but also for the light they shed on the production and transformation of concepts.

The latter remark relates to an important third issue. As this case study will show, even though a correlation between concepts and cultures can be demonstrated, concepts are also not static. The range of equations covered by the successive concepts increases. The understandings of equations they reflect, as we perceive it from a present-day viewpoint, deepen. Algorithms for solving equations gain in generality. The cultural considerations I find useful to introduce thus do not lead to a static vision: in parallel with the changing cultural contexts, the concept of equation undergoes transformation. Nor do they lead to a view of cultures as bounded wholes. This issue relates to my last thesis.

It is true that the transformations in the concept of equation that can be perceived in my corpus occur within a specific tradition. All these concepts are members of a clearly defined subfamily in a larger family of equations evidenced by a corpus of sources originating from many places in the world. But nowhere else in the world do we find the way of conceiving of and handling equations to which these Chinese sources attest. The characteristic features of this subfamily of concepts can be correlated with continuities displayed in the ways of carrying out mathematical activity in the tradition of ancient China considered. Yet, this does not mean that no concept from this subfamily could be appropriated elsewhere. My conclusion mentions what appears to be a sudden occurrence of this approach to equations within Arabic sources of the twelfth century.

In brief, scholarly cultures as I see them capture specificities of ways of working in a given collective. They are by no means isolated, bounded, and unchanging cells. Moreover, concepts are not buried in cultures. Even though they display adherence to scholarly cultures in which they took shape, they circulate and can be appropriated in other cultural contexts. Through such a process, universality is constructed.

These are the theses this chapter illustrates. The argument is developed in three sections. First, I outline what I mean by "scholarly cultures" and sketch features attesting to the radical change I detect in practices of mathematics in China between the first and the thirteenth century. The second, longest section focuses on the first period, spanning the time from the first to the seventh century, and argues that we can identify a concept of quadratic equation in the earliest sources considered. I show how taking cultures into account helps us to interpret the sources and highlight the correlation between a given mathematical culture and a concept. The third section deals with a subsequent time span, ranging from the eleventh to the thirteenth century and in fact beyond. It outlines a subsequent culture and a subsequent concept of equation. Both display continuities and differences with those described in the second section. In conclusion, I return to the theses expounded at the beginning of the chapter.

Mathematical Cultures and Our Sources

One of the difficulties attached to the exercise of interpretation has already been the object of much discussion.[2] It relates to the practice that involves using modern scientific concepts to read ancient documents. I think this is a necessary step, but only a first step if we are to avoid conceptual anachronism.[3] This chapter examines another form of anachronism, which consists of approaching our sources from the viewpoint of modern textual categories and reading mathematical problems, figures, algorithms, inscriptions for computing, and so on, as their modern counterparts. My point is that the practices through which actors engaged with problems, algorithms, and more generally what was written down or inscribed in the performance of a given mathematical activity—practices that I shall designate as "elementary practices" or "elements of practice"—determine at least partly the meaning of these textual elements. Such elements of practice cannot be taken for granted a priori and need to be described. Their set constitutes essential components of the mathematical cultures I am trying to comprehend. It is at this juncture that the issue of the cultures within which our actors operated connects with that of interpretation.

Concretely, how are we to proceed, since the sources we want to interpret are, in fact, our main vehicles for carrying out this task? To start with, our sources contain many hints indicating how actors dealt with the various kinds of textual elements composing the sources—such as problems and algorithms. We can collect these hints and rely on them to describe which practices that actors shaped and shared with the textual elements leave a substantial trace in our sources. For the Chinese sources considered here, this procedure shows the elementary practices in question are specific to a given context. Moreover, in that context, these elementary practices had specific connections with one another. This nexus of practices reflects a specific organization of mathematical activity and provides a first sketch of the mathematical cultures I aim to describe.

Our sources were more generally produced within a material environment where other objects could be used to carry out mathematical activity. The material objects of the mathematical culture of the first time period under study in ancient China included an instrument for computing and blocks to practice spatial geometry. Our sources also provide evidence about some of their material features and the ways in which these features were employed. The reason is simple: because our sources were produced in the context of an activity that involved elements of text and material objects, they bear marks that derive from this contiguity and reflect the practice in general. Like above, the practices with these material objects are also components of the cultures, as I view them. The same applies to the link between these practices and the other elements of practice. Describing these practices is part of our task. In the study presented here, it proves all the more necessary to deal with these material objects that we encounter a case where inscriptions that are essential to the story and once lay outside the writings became integrated into the texts at a later date. To be able to consider a long-term conceptual history, we thus need to take into account a corpus of texts and inscriptions that present different material features at different time periods.

Let me first illustrate the abstract description just outlined, using concrete examples. The earliest extant Chinese documents attesting to a concept of quadratic equation are books dating from about the beginning of the Common Era and handed down through the written tradition. By contrast, as far as I know, none of the mathematical documents yielded by archeological excavations deal with this topic.

The earliest extant book in which we can identify a concept of quadratic equation is *The Nine Chapters on Mathematical Procedures*, whose composition I date from the first century CE. Like the other early mathematical book handed down, *The Gnomon of the Zhou*, which dates from roughly the same time period and is connected with the practice of astronomy, *The Nine Chapters* was apparently perceived as canonical shortly after its completion. Accordingly, commentaries were composed on both books. These commentaries are essential documents for describing ancient practices in mathematics. The earliest extant commentary on *The Nine Chapters* was completed by Liu Hui in 263, and in the same century Zhao Shuang authored the earliest known commentary composed on *The Gnomon of the Zhou*. Both commentaries were handed down with their respective canons.[4]

The practice of mathematics to which *The Nine Chapters* attests employs three key elements. The book is composed of problems and algorithms. The texts of the algorithms further refer to a material object, outside the text, that is, an instrument with which practitioners computed. This instrument was composed of a surface, the material features of which we can only speculate about, and also of counting rods placed on that surface and used to represent numbers according to a decimal place-value number system.[5] The canon and the commentaries provide hints on the practices with these three elements. In the absence of more substantial descriptions of these practices by the actors themselves, we can rely on these hints to show that the practices in question are quite specific and certainly different from our own practices with similar elements.

The commentators' practice of mathematics attests to a richer set of elements.[6] Commentaries systematically include proofs of the correctness of the algorithms contained in the canon, as well as second-order discussions about various facets of mathematics. Commentaries also contain references to tools of visualization, which were absent from the canons: diagrams for plane geometry and blocks for space geometry. I have argued that at the time, diagrams were, like blocks and counting rods, material objects outside the text (Chemla 2001). The commentaries give information on material features of these objects and practices with them. The same conclusion as above holds true: all these practices differ from our expectations. In brief, these sources testify to a practice of mathematics for which writings contain only discourse (problem, algorithms, proofs, discussions, etc.), whereas all the other elements (counting rods, diagrams, and blocks) are material objects used in conjunction with texts. This sketch explains why the usual appearance of a page in our sources for this first time period looks like what is shown in figure 14.1. The page contains nothing but characters.

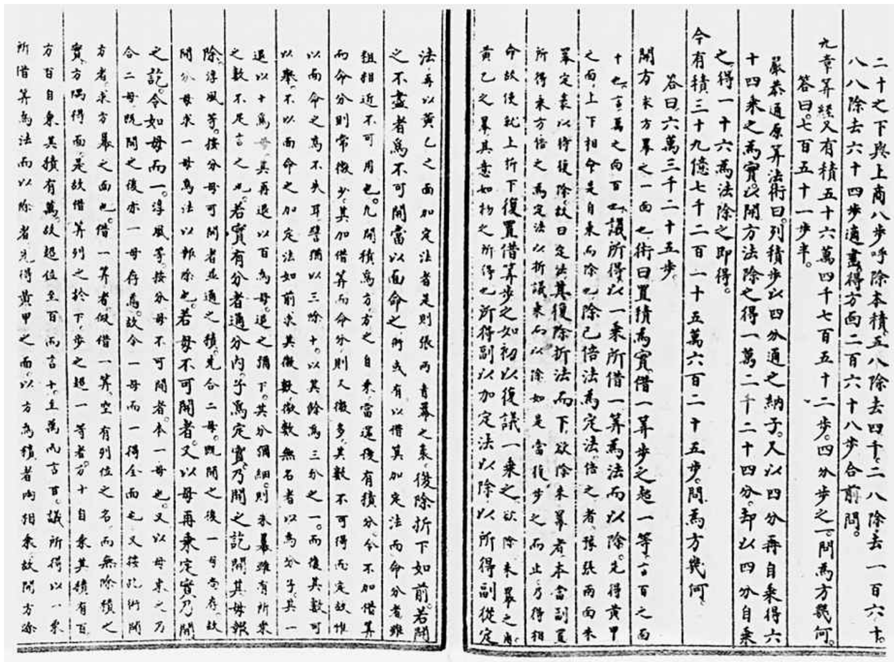


FIGURE 14.1. A page from *The Nine Chapters* with its earliest extant commentaries, as reproduced in the fifteenth-century encyclopedia *Grand Classic of the Yongle Period* (*Yongle dadian*), 1408, chapter 16344, 9b–10a.

By contrast, the aspect of the sources handed down from, roughly speaking, the tenth century onward attests to a radical change in the practice of mathematics.[7] I limit myself to sources that matter for the discussion here, even though the general features described hold more broadly (Chemla 2001). These sources include remaining chapters of a thirteenth-century subcommentary on *The Nine Chapters* and Liu Hui's commentary, which Yang Hui completed in 1261 (Lam 1969; Yan 1966). The corpus also includes a book written in 1275 also by Yang Hui, *Quick Methods for Multiplication and Division for the Surfaces of the Fields and Analogous Problems* 田畝比類乘除捷法. In particular, I am interested in the last part of its final chapter, in which Yang quotes at length a book that probably dates from the eleventh century, *Discussing the Source of the Ancient (Methods)* 議古根源 by Liu Yi 劉益.[8]

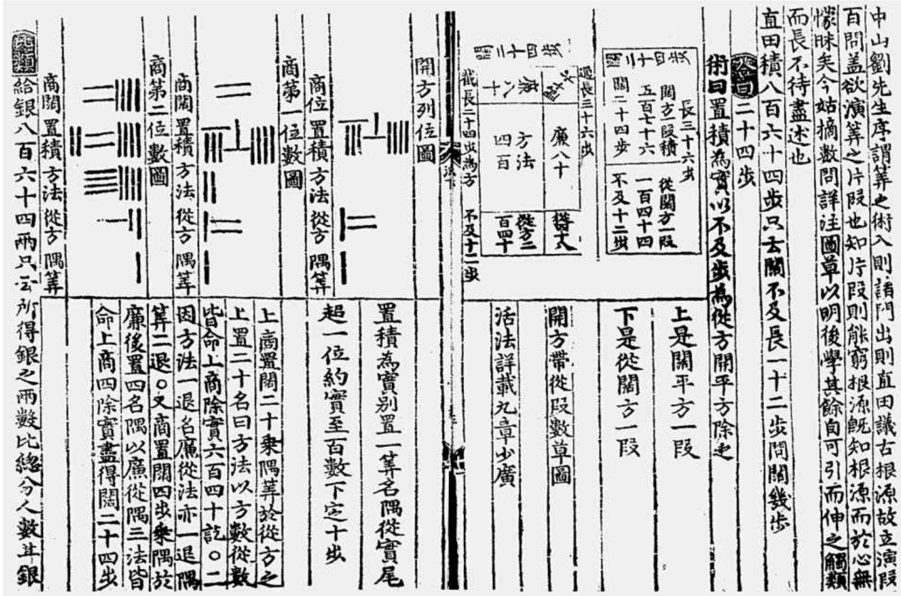


FIGURE 14.2. A page from the quotation of Liu Yi's *Discussing the Source*, in Yang Hui's *Quick Methods for Multiplication and Division for the Surfaces of the Fields and Analogous Problems*, Korean edition from 1433, reprinted in Kodama 1966 (91).

Figure 14.2 shows a page of *Discussing the Source* as quoted by Yang, which vividly illustrates the radical change evoked. The page combines discourse with geometrical figures, on the right-hand side, and illustrations of configurations of numbers represented with counting rods, on the left-hand side. The discourse still contains problems, algorithms, and proofs of their correctness. It still refers to a surface outside the text on which computations were carried out. The key point is that in the eleventh century at the latest, pages of books included new nondiscursive elements. Their inclusion within books goes together with a shift in the practices with them.[9] More generally, the mathematical culture in relation to which these writings were composed presents continuities with and differences from the earlier one mentioned above. Both the continuities and the differences are important, as I show in discussing quadratic equations.

What are the consequences of these remarks for understanding equations and the facets of their history that these Chinese sources allow us to perceive? First, only by taking practices with problems, algorithms, the computing instrument, proofs, and diagrams into account can we grasp the concept of quadratic equation attested to in writings from the first time period mentioned above. On this basis, correlations can be established between this concept and the practices attached to it. Second, understanding the deep transformations in mathematical practice that occurred probably in the tenth or the eleventh century is essential for capturing the conceptual and material continuities that, despite crucial differences, tie the concept of equation in the first time period to that of the second. Against this backdrop of continuity, we can perceive key changes in the concept of and practices with equations.

This sketch shows clearly that we face a methodological problem if we want to address questions of diachrony. Whether we want to describe the earliest concept of equation evidenced or to appreciate its similarities with and differences from later concepts of and practices with equations, we must restore practices to which our sources refer but that in the first time period left no material traces in the writings. The following section addresses this issue.

A Culture and a Concept of Quadratic Equation in the First Centuries CE

Chapter 9 of *The Nine Chapters* is devoted to the right-angled triangle. Problem 19 in that chapter is the only problem in the book to be solved by a quadratic equation. As usual in *The Nine Chapters*, this problem describes a specific setting, which I represented in figure 14.3. Note that this diagram, which I drew to help the reader follow the argument, corresponds to nothing in the sources. Nevertheless, I followed Chinese conventions that place north downward. The problem introduces a situation and particular numerical data, requiring the determination of an unknown quantity. Hints gleaned in Liu Hui's commentary show that such a problem with the procedure solving it was read as a general statement, although its formulation was not abstract (Chemla 2003). This gives a first example of the connection between the description of practices and the interpretation of a text.

Here is an outline of the problem in question. The length x of the sides of a square town, whose walls face north-south and east-west, is unknown. Someone leaves the town through its southern gate and walks a distance s. (Note that s is my notation for what in the text is a numerical value, expressed with respect to the measuring unit for length *bu* [step]. The same convention holds below.) At the distance s southward, the walker turns westward and, after walking a distance w, sights a tree, which is northward at the distance n from the northern gate. The problem asks for the length of the sides of the town.

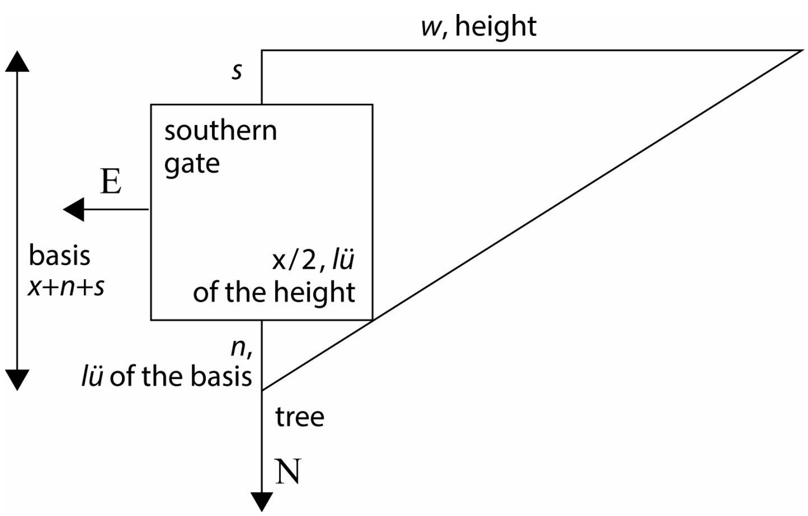


FIGURE 14.3. An illustration of problem 19 of *The Nine Chapters*

The Nine Chapters contains a procedure for solving this problem. Its text constitutes the earliest extant piece of evidence documenting quadratic equations, and the only one dating from the first century at the latest.[10] The challenge its interpretation poses is clear, when one examines the text, which I translate as follows:

> Procedure:
>
> One multiplies, by the quantity of *bu* outside the northern gate, the quantity of *bu* walked westward, and one doubles this, which makes the **dividend**.
>
> One adds up the quantities of *bu* outside the southern gate and northern gate, which makes the **joined divisor**.
>
> One **divides [the dividend] by this by extraction of the square root**, which gives the side of the square town.[11]

Let us observe the main features of this text. It refers to a "procedure." The text of the procedure brings into play names of operations: multiplying, doubling, adding up, and so on. It consists of a list of operations whose execution, the text asserts, yields the unknown in question. The text thereby attests to the fact that some operations have been identified as key operations (multiplication, duplication, addition, and so on). They are the building blocks of any algorithm. Identifying key operations is one of the main theoretical goals of mathematical work in the scholarly culture observed (Chemla 2010a). The text presupposes that the reader knows algorithms for executing these operations.

An operation bears on operands (e.g., a multiplication operates on two values). During that time period in China, specific terms were introduced only for the operands of division. I translate these terms as "dividend" and "divisor." Division has another specificity. As I show below, through various kinds of practices other key operations are shown to be analogous to it. The terminology for operations mirrors this fact, as shown by the text of the procedure quoted above. The prescription of a square root extraction reads "one *divides* … by extraction of the square root," indicating a relationship between root extraction and division. The same feature holds true for the terms designating the operands. Similar operations have similar operands. Whereas a division operates on two operands, a dividend and a divisor, the root extraction operates on only one: the number whose root is sought.[12] That operand is called a dividend. A network of relationships is thereby established between key operations, and the terminology reflects this.

These elements of information allow me to highlight an essential feature of the text quoted above. I have marked its structure using bold characters. It prescribes the computation of two values that are taken, respectively, as "dividend" ($2nw$) and "joined divisor" ($s + n$); following this, it prescribes: "divide … by extraction of the square root." This structure shows that a higher-level operation, related to root extraction but having two operands, concludes the procedure. It is this higher-level operation that I suggest interpreting as a kind of quadratic equation—I call it the "operation-equation."

Indeed, no modern commentator denies that the last operation of this procedure is equivalent to the quadratic equation that we would write today as[13]

$$2nw = (s + n)x + x^2.$$

Such a retrospective reading can, however, represent only the first step in the practice of conceptual history. In this chapter, I am actually interested in the problems raised by the interpretation of the text: In which sense does this operation correspond to a quadratic equation (i.e., which concept of equation do we have)? How can one argue for this interpretation? How did the ancient reader achieve this understanding? Since nothing else is added to this procedure, how did the ancient reader know how to execute the operation (i.e., to solve the "equation")? How was the operation (i.e., the equation) established? I claim that further description of the mathematical culture, in the context of which this text was written, offers clues to argue for an answer to each of these questions, and describe how practitioners worked with equations conceived in this way. The arguments outlined below aim to illustrate more generally the relationship between the issue of scholarly cultures and that of interpretation.

The Establishment of a "Quadratic Equation" and Elements of Practice with Diagrams

The procedure quoted above first prescribes the computation of two operands—$2nw$ is the "dividend" and $(s + n)$ is the "joined divisor"[14]—and then prescribes a final operation as a "square root extraction." How should we interpret the latter prescription? To begin with, let me outline, in modern terms, how the third-century commentator comments on this procedure.

Liu Hui introduces two similar right-angled triangles (see figure 14.4). He says that one of these triangles has the path described westward, w, as its "height" and the distance between the tree and the southernmost point reached by the walker as its "base." The base amounts to $n + x + s$. The second triangle has the distance northward n as its base, while half the side of the town constitutes its height ($x/2$). Note that all quantities are related to lines identified with reference to the actual geometrical situation. This holds true for the whole commentary. By means of a rule of three, the similarity between the triangles leads to the equality $(n + x + s)\, x/2 = nw$. What is essential is that Liu Hui refers to this equality as holding between areas again by reference to the situation on the field. The value nw measures the area of the horizontal rectangle, in the lower part of figure 14.4. It is equal to $(n + x + s).x/2$, which, Liu Hui says, "occupies the half to the west." If one relies on figure 14.4, which sketches a cartographic view of the situation, it is clear that his statement refers here to the vertical white rectangle that covers the western half of the city and extends beyond to the north and the south. Liu Hui goes on: "If, further, one doubles this, one adds the eastern (part) to it, which exhausts it (the area of the rectangle) entirely." The operation of doubling yields numerically $2nw$, and geometrically it adds to the white vertical rectangle the rectangle I represent in gray on figure 14.4. Note that the commentator has thereby introduced a (vertical) rectangle, the area of which is precisely the dividend ($2nw$) yielded by the first two operations in the procedure of *The Nine Chapters*. In fact, the term *shi*, which I translate as "dividend," also means "area." I argue below that both meanings are active here.

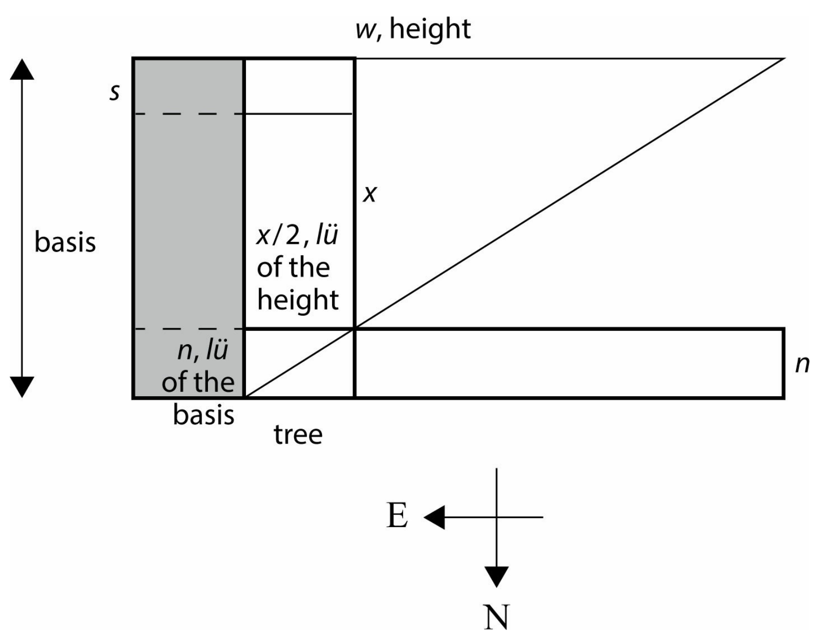


FIGURE 14.4. The geometric reasoning developed by Liu Hui to account for the correctness of the procedure solving problem 19. The text refers only to areas, not to a diagram.

Graphically, on the basis of the data (n, s, w), one has determined the area of a rectangle composed of a square, whose side is the unknown, and two rectangles north and south of it (figure 14.5a). This is what Liu Hui recapitulates in the subsequent sentences before alluding to a reshaping of this rectangle. His commentary reads (I invite the reader to rely on figure 14.5 to understand the commentator's reasoning and to wait patiently for the elements of the passage that are still obscure):

> The area of this procedure is the area that, from east to west, is like the side of the square town and, from north to south, goes from the tree up to the end of the 14 *bu*, to the south of the town [i.e., from the northernmost to the southernmost point; see figure 14.5a]. Each of the (amounts of) *bu* north and south makes a width, and the side of the town makes the length, this is why one places the two widths side by side to make the joined divisor [see figure 14.5b]. The sum (of their areas) is taken as the area outside the corner.

This passage is extremely rich in information. A few remarks on how the text describes graphical operations will be helpful. In the tradition of writing about mathematics to which the text belongs, the designation of a length (north-south direction) and a width (east-west direction) is the usual way to point out a rectangle. Here, Liu Hui draws attention to two rectangles, one in the north and the other in the south. His last statement explains how they are brought together graphically and how the global rectangle, whose area is known, is reshaped. The result of this operation is illustrated by figure 14.5b. Such handling of diagrams as material objects is typical of the way in which this type of visualization device was used in the mathematical culture evidenced by Liu Hui's commentary (Chemla 2010b). The absence of anything visual in the text fits with the main trends regarding the composition of writings in the context of this mathematical culture, outlined in the preceding section. These remarks constitute the first features of the practice with figures we encounter.

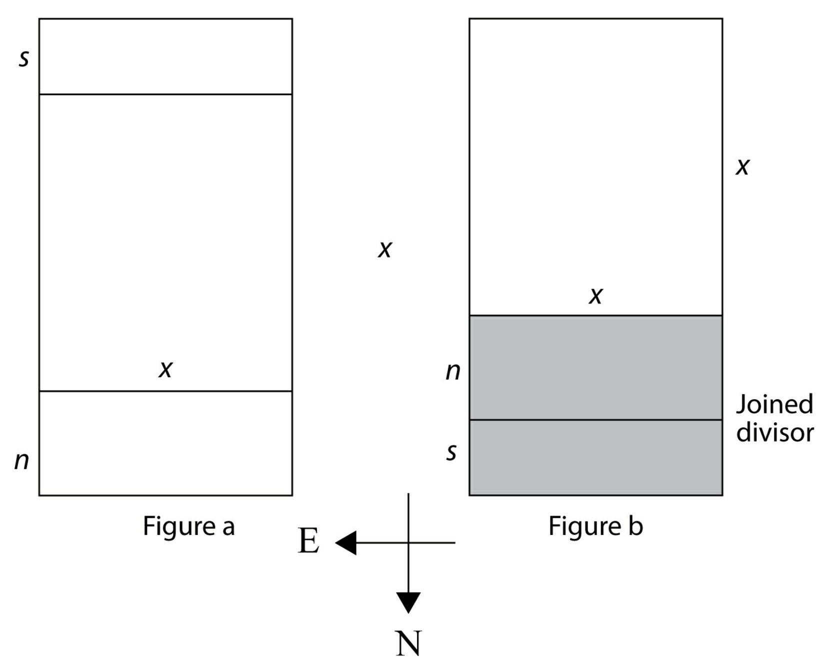


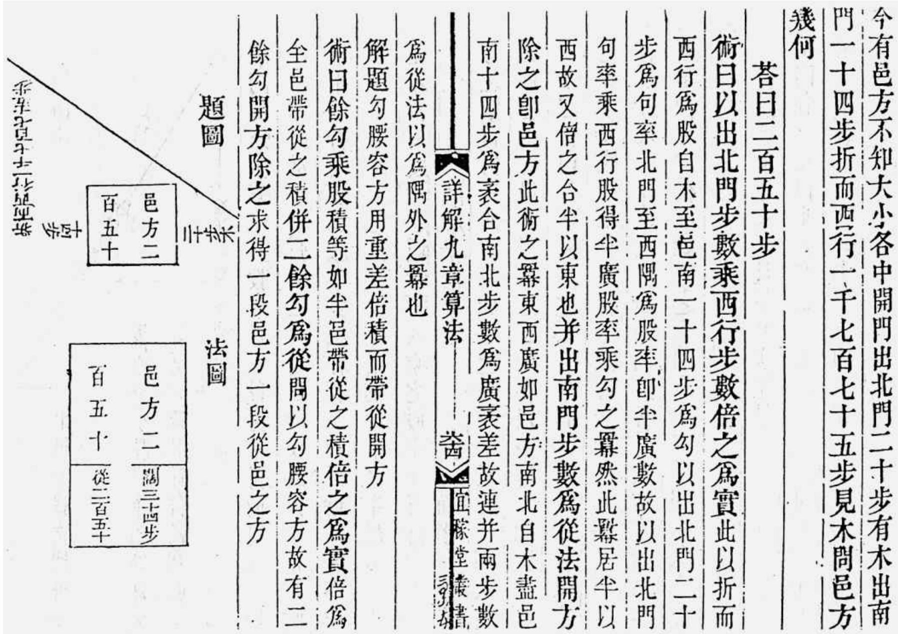
FIGURE 14.5. Liu Hui's establishment of the "operation-equation" solving problem 19 of *The Nine Chapters*: (a) the area corresponding to the dividend of the procedure. (b) the side corresponding to the joined divisor of the procedure.

The statement referring to the key graphical operation is crucial. Liu Hui writes: "*this is why* one *places* the two widths *side by side to make the joined divisor*" (my emphasis). The positioning of the two small rectangles together forms a rectangle of width $n + s$. This addition is precisely the third operation prescribed by the procedure in *The Nine Chapters*, which determines the value of the joined divisor. Liu Hui's use of the particle *gu* ("this is why") at this juncture indicates he understands the graphical transformation as accounting for the meaning of the computation of the joined divisor. At this point, the commentator has established that a rectangle whose area is $2nw$ combines a square of side x and a rectangle with sides, respectively, equal to x and $n + s$. In modern terms, we could write what he has shown graphically as

the area of the rectangle (the dividend $2nw$) equals $x^2 + (n + s).x$,
where he takes $(n + s)$ to be the joined divisor.

These statements conclude his commentary. The description of Liu Hui's practice of commentary shows that, in his view, he has accounted for the correctness of the procedure of *The Nine Chapters*. This interpretation implies that the establishment of the rectangle and of its structure accounts for the correctness of using the final operation of the procedure. In other terms, the rectangle and its structure provide the "meaning" of the operation in which we are interested. Accordingly, for Liu Hui, inasmuch as the rectangle writes the equality sketched above, the final operation of the procedure does correspond, as all historians interpreted it, to a quadratic equation.

The conclusion just obtained uses the commentary to establish what Liu Hui understands is the meaning of the operation-equation. We can also look at the commentary from another angle and ask what it tells us about the concept of equation as Liu Hui understands and practices it. In this commentary, Liu Hui establishes the equation in the sense that he establishes the correctness of using the operation-equation, as the procedure shapes it, to solve the problem. He does so by establishing a rectangle that brings the area $2nw$, the joined divisor $(n + s)$, and the unknown into relation. The equation is thus yielded graphically, its figure being that of a rectangle of known area a that can be broken down into a square x^2 and a rectangle bx. The term a corresponds to what *The Nine Chapters* designates as dividend, and b is the joined divisor. These facts are actually not contingent. Every time we meet an operation-equation of this type in an algorithm stated by Liu Hui or Zhao Shuang, it can be interpreted as deriving from the reading of a similar rectangle in the geometrical situation considered (Chemla 1994). The graphical writing of the operation-equation corresponds, in this context, to what we would call "equality" today. This form of inscription plays an essential part in my narrative—I return to this point later. Let me refer to this graphic formula as the first facet of the concept of equation attested to in the first time period.[15] It is all the more important to stress this facet that these diagrams do not appear explicitly in the early sources. Overcoming this difficulty was one of the methodological challenges identified in the first section. I have shown how to solve this difficulty by exploiting hints gleaned from Liu Hui's commentary. The reconstruction is also supported by later evidence. In Yang Hui's thirteenth-century subcommentary on *The Nine Chapters*, composed in the context of a mathematical culture in which writings include illustrations, a diagram illustrating the situation of problem 19 and a diagram showing the rectangle writing the equation were inserted in the pages of the book (see figure 14.6).


今有邑方不知大小各中開門出北門二十步有木出南門一十四步折而西行一千七百七十五步見木問邑方幾何

荅曰二百五十步

術曰以出北門步數乘西行步數倍之爲實此以折而西行爲股自木至邑南一十四步爲勾以出北門二十步爲句率北門至西隅爲股率即半廣數故以出北門句率乘西行股得半廣股率乘勾之冪然此冪居半以西故又倍之合半以東也并出南門步數爲從法開方除之即邑方此術之冪東西廣如邑方南北自木盡邑南十四步爲袤合南北步數爲廣袤差故連并兩步數爲從法以爲隅外之冪也

解題勾股容方用重差倍積而帶從開方

術曰餘勾乘股積等如半邑帶從之積倍之爲實倍爲全邑帶從之積併二餘勾爲從隅以勾股容方故有二餘勾開方除之求得一段邑方一段從邑之方

題圖

法圖

FIGURE 14.6. Yang Hui's thirteenth-century subcommentary on problem 19 illustrates the situation described and the operation-equation solving it (Yang Hui 1261, edition from the *Yijiatang congshu*, 1842, 64).

Liu Hui establishes the correctness of using the operation-equation but adds nothing on its execution, nor is there any explicit treatment of this question anywhere in the canon or in the commentary. And yet the equation is meant to be solved, since the outline of problem 19 is followed by an answer. Other puzzles remain. Why are *a* and *b*, respectively, designated as "dividend" and "joined divisor"? Why does the commentary refer to the rectangle *bx* as "area outside the corner"? Why is the solution of the equation prescribed using the expression "one divides … by extraction of the square root"? In fact, all these questions find an answer through a specific interpretation of the algorithm for extracting square roots that *The Nine Chapters* provides in chapter 4. I outline this algorithm next, which leads to a second facet of the quadratic equation and to the second methodological challenge stated in the first section of this chapter.

The Extraction of a Square Root, and Practices with Algorithms and Computations

The Nine Chapters contains the text of an algorithm to extract square roots. Its interpretation requires the description of two elementary practices. First, we need to analyze how texts for algorithms were written down in the mathematical culture to which *The Nine Chapters* and its commentaries testify. Second, we must reconstruct how the instrument for computing, that is, a surface on which numbers were represented with counting rods, was used at the time.[16]

As is the case for many other writings composed in China between the first and the thirteenth century, the text of the algorithm describes the process of root extraction by reference to another process of computation, that is, as if it were a division. This remark accounts for how the text employs the technical terms attached to the execution of a division (dividend, divisor, quotient, eliminate, move forward, move backward). It also proves essential in helping to interpret the text for square root extraction. In this way, the text further shapes an analogy between the two processes of computation of division and root extraction, showing in which respect a square root extraction is a kind of division. A statement is asserted in the way the texts display similarities and differences between the algorithms. Interpreting the text on root extraction by relying on how it refers to a process of division requires reading this analogy.

The operation corresponding in *The Nine Chapters* to a quadratic equation has two operands (a dividend and a joined divisor), whereas its execution is prescribed as a square root extraction. In the same way as the prescription of square root extraction refers to division, that of the operation-equation refers the solution of the equation to another operation, that of root extraction. Here too, it proves essential to rely on this statement of a relationship to interpret the text on which we concentrate. In the cultures considered, relationships between operations were regularly expressed using related terms to prescribe them and also describing their processes of computation in related ways. To interpret such texts and grasp the analogy they formulated, the reader apparently needed to use the reference that a text of procedure made to other texts. This practice of intertextuality is a feature characterizing the production and interpretation of texts for algorithms in the mathematical cultures under study. It echoes other features characteristic of the practice with the tool of computation. Let me now consider them.

Some of the technical terms used for division and taken up in the text of the algorithm for square root extraction designate elementary operations on the surface for computations: "moving forward" means to take a number written down with rods, using a place-value decimal system, and move it column by column leftward. Its value is thus multiplied by 10 each time it is shifted by a column. "Moving backward" refers to the reverse movement. These elementary operations exploit properties that rods lend to the representation of numbers. Numbers expressed with rods can be moved on the surface on which they are written, and their value can be modified and replaced by another value. In the text for root extraction, the operations of moving backward and moving forward are applied to numbers placed in the position of "divisors," whose shifts on the surface appear to be similar to those of a divisor in a division. More generally, the practice, outlined above, of expressing relationships between high-level operations (division, root extraction) through shaping a relationship between the texts of the algorithms that carry them out echoes a practice of expressing a relationship between operations through shaping their processes of computation on the surface in a similar way.

To shed light on this practice at the time when *The Nine Chapters* was written, we must reconstruct how the surface was used for computations, since at that time no illustrations or even generally no descriptions were included in the writings. Knowing more about the handling of computations nevertheless proves essential for answering the questions about quadratic equations raised above. This represents the second methodological challenge. Recovering ancient practices of computation also relies on hints gleaned from texts. Like "moving forward" and "moving backward," some of the terms the texts use reveal material features of the surface and its use. In this case too, results can be compared with the evidence contained in later subcommentaries on *The Nine Chapters*, composed at a time when writings included illustrations. For instance, Yang composed illustrations for his subcommentary on *The Nine Chapters*, in which he presented a cognate but different algorithm to extract square roots. His illustrations show successive moments in the computation on the surface, through arrays of numbers written with counting rods. These arrays are similar, in their material features, to what the texts themselves allow us to reconstruct about the earlier practices. Let us consider some of the features that are important here.

Computations on the surface make a crucial use of positions. The execution of a division is performed on positions arranged in three lines: the dividend in the middle row, the divisor in the row below, and in the upper row, the quotient obtained digit by digit. The fact that the text of the algorithm for root extraction uses these same three terms is correlated with the fact that the execution of the algorithm also develops fundamentally on three positions: the number whose root is sought is placed in the middle row, whereas the root is determined digit by digit, the successive digits being placed in the upper row, as in a division. As for the row below, numbers are gradually shaped throughout the process of computation so that the overall scheme and elementary operations of the root extraction can be correlated with those of a division. Accordingly, the numbers that succeed to each other in the position under that of the dividend are called "divisor."[17]

Positions of that kind are the key components of any operation considered in ancient Chinese documents. Texts for algorithms attribute names to them. The values placed in these positions change while a computation is executed. At the point of a text when a term designating a position is used, the computation involving this position picks up the value placed there at the time.

These are the first elements of a description of the practice of computation specific to the mathematical culture considered. Its key feature for the aim pursued here is that throughout the computation, the inscription of the process executing a square root extraction demonstrates that the positions named "dividend," "quotient," and "divisor" record the same events as the homonymous positions in a division. The processes of computation are thereby shown to combine the same elementary patterns of operation, which the positions allow us to grasp. In that way, a dynamic relationship is shaped between different processes of computation. Similar dynamic relationships are evidenced in various Chinese writings, composed at different time periods. One can also show that these dynamic relationships are the object of a mathematical work, since they were regularly rewritten to display the relationship in a new way.

These facts suggest that the dynamic inscription of the process on the surface was meaningful for the actors and was read as such. It is this dynamic inscription that the illustrations by Yang attempt to capture within the pages of a book. My own recovery of the practice of computation leads me in a similar way to reconstruct the successive states of the surface throughout a root extraction according to the algorithm in *The Nine Chapters*. In tables 14.1 and 14.2, I display the process as I restore it—I first use Arabic numerals instead of numerals represented with rods. Since my only intention is to provide the reader with a visual aid for the subsequent discussion, there is no need here to attempt to understand the computations.

TABLE 14.1. The First Sequence of Computations in a Square Root Extraction in *The Nine Chapters*

Step 1	Step 2	Step 3	Step 4	Step 5	Step 6	(Steps)
			2	2	2	Upper: quotient
55225	55225	55225	55225	55225	15225	Middle: dividend
1	1	1	1	2	2	Lower: divisor

The related text of the procedure reads: "[Step 1] one puts the number-product as dividend. [steps 1 to 3] Borrowing one rod, one moves it forward, jumping one column. [step 4] The quotient being obtained, [step 5] with it, one multiplies the borrowed rod, 1, once, which makes the divisor; [step 6] then with this, one eliminates." The terms underlined are common with the process of division.

The process of extracting the square root of 55225 begins as shown in table 14.1. It suffices for our purpose to represent the subsequent steps of the root extraction symbolically. In modern mathematical notation, if A is the number whose root is sought, and if $a\,.\,10^n$ and $b\,.\,10^{n-1}$ are the first two digits of the root, the subsequent computations can be represented as shown in table 14.2. (I return below to the meaning of the computations.)

Why do these computations determine a square root, and what is the connection with understanding the nature of the quadratic equation as it occurs in *The Nine Chapters*, and how it was solved? These questions, as well as those raised above, are answered when we consider how Liu Hui fulfills the task of proving the correctness of the above-mentioned algorithm. Through sketching his reasoning, we can elucidate the meaning of the computations displayed above. Indeed, this elucidation is the crux of my argument.

The Practice of Proving the Correctness of Algorithms and a Second Facet of "Equations"

As I mentioned above, for all procedures the commentaries on *The Nine Chapters* systematically address the question of their correctness. This holds true also for the algorithm to extract square roots. In general, the commentators use a *dispositif* with which they can explicitly show "the intention" of the successive operations prescribed by the algorithm. The intention (*yi* 意) of an operation or a subprocedure is the meaning of its result, formulated with respect to the dispositif. At the end of the reasoning, the meaning of the final result of the algorithm is established and shown to conform to what was expected. These features partly characterize the practice of proving the correctness of algorithms in the mathematical culture examined in this whole section.

TABLE 14.2. The Subsequent Sequence of Computations of the Square Root Extraction in Modern Terms

Step 6	Step 7	Steps 8 and 9	Step 10	Position
$a.10^n$	$a.10^n$	$a.10^n + b.10^{n-1}$	$a.10^n + b.10^{n-1}$	Quotient
$A-(a.10^n)^2$	$A-(a.10^n)^2$	$A-(a.10^n)^2$	$A-(a.10^n)^2$	Dividend
$a.10^{2n}$	$2a.10^{2n}$	$2a.10^{2n-1}$	$2a.10^{2n-1}$	Divisor
			1	Below: auxiliary

Step 11	Step 12	Step 15	Position
$a.10^n + b.10^{n-1}$	$a.10^n + b.10^{n-1}$	$a.10^n + b.10^{n-1}$	Upper row: quotient
$A-(a.10^n)^2$	$A-(a.10^n)^2$	$A-(a10^n + b10^{n-1})^2$	Middle row: dividend
$2a.10^{2n-1}$	$2a.10^{2n-1}$	$2(a.10^{2n-1} + b.10^{2(n-1)})$	Lower row: divisor
$(10^{n-1})^2$	$b\,.(10^{n-1})^2$		Below: auxiliary

The related text of procedure in *The Nine Chapters* reads: "[Step 6] After having eliminated, [step 7] one doubles the divisor, which gives the fixed divisor. [steps 8, 9] If again one eliminates, one reduces the divisor, moving it backward. [step 10] Again, one puts a borrowed rod; [step 11] one moves it forward as at the beginning; [step 12] with the next quotient, one multiplies it once." The terms underlined are common with the process of division. Steps 12 and 15, related to the digit b, set the stage for the following digit. (Steps 13 and 14 are omitted as I only reproduce the initial and the final stage.)

For square root extraction, Liu Hui establishes the meaning of each operation or group of operations by reference to a single visual device. The various hints his commentary gives enable us to restore its structural and material features, as shown in figure 14.7. Within the square of area A, Liu Hui identifies three types of areas that are essential for his proof and to which he attributes three different colors. First, he distinguishes three squares, in yellow. They represent the squares of the successive digits with the respective order of magnitude. Second, Liu Hui distinguishes two sets of rectangles, the first in vermilion and the second in blue-green. He uses these colors to refer to the tinted rectangles and to explain what the successive steps of the algorithm compute.[18] In this case too, the main structural features of the diagram conform to diagrams Yang inserts in his subcommentary. These hints reveal continuities in the practice of visual devices between the first and the second time periods considered here, beyond the break represented by the insertion of illustrations within books from about the tenth century onward.

I shall mention only some aspects of Liu Hui's proof. The part played by the diagram in the proof will turn out to have an essential role in our story. Liu Hui interprets steps 1–6 (table 14.1) as aiming to subtract from the area A—which is placed in the dividend row—the area of the yellow square whose side is $a\,.10^n$. When this area is subtracted, the remaining "dividend" or "area," equal to $A - (a\,.10^n)^2$, has the shape of a gnomon. The determination of the subsequent digits of the root amounts to finding the value of the width of the gnomon. Liu Hui interprets steps 7 and 8 (table 14.2) as aiming to prepare, in the divisor row, what corresponds to the length of the two vermillion rectangles. (Note that $2a.10^{2n-1}$ corresponds precisely to twice $a\,.10^n$ multiplied by 10^{n-1}, that is, the order of magnitude of the second digit. It suffices to multiply this value by b to obtain the areas of the two rectangles.) Further, Liu Hui interprets steps 10–12 as preparing, in the row below, what needs to be multiplied by b to yield the area of the second yellow square on the diagram. The sum of these two values is placed in the divisor row in step 13, and its multiplication by b yields the area of the gnomon of width $b\,.10^{n-1}$, which needs to be subtracted from the area of the overall remaining gnomon to deal with the digit b. Thereafter, the following computations (until step 15) are interpreted as determining similarly the length of the two blue-green rectangles to prepare a similar treatment for the subsequent digit.

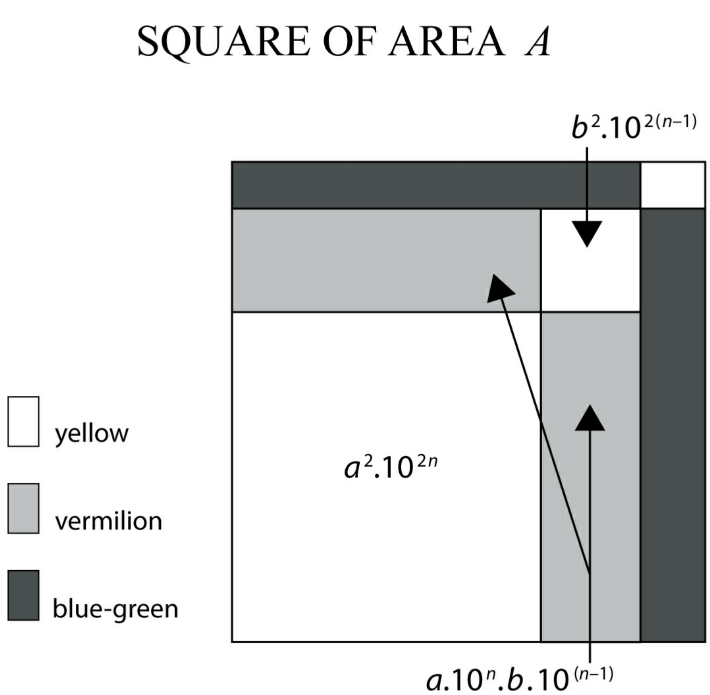


FIGURE 14.7. Restoring Liu Hui's visual device for the proof of the correctness of the root extraction.

Liu Hui's interpretation of the computations with respect to the diagram sheds light on a fact essential for our purpose. Once the first digit is dealt with and the length of the two vermilion rectangles prepared, continuing with the root extraction is equivalent to determining the width of a gnomon having an area equal to $A - (a.\ 10^n)^2$. Thus, if we forget about the first digit and concentrate on the gnomon, the part of the algorithm that starts at step 8 (table 14.3) is an operation that yields, as a result, the width of the gnomon. The same reasoning holds true in the step following step 15 (step 16 in table 14.3): if we subtract the larger square, which contains the first two yellow squares and the two vermilion rectangles, the continuation of the algorithm yields the width of the gnomon with blue-green rectangles. Continuing the root extraction by starting at either of these moments in the execution of the algorithm, and forgetting about the first part of the root determined, amounts to operating on an array similar to that shown in table 14.3.

TABLE 14.3. The Steps of the Square Root Extraction in Which One Can Forget About the Digits So Far Computed and Nevertheless Go On with the Operation and Determine the Width of a Gnomon

Step 8	Step 16	Position
		Upper: quotient
$A - (a\ .10^n)^2$	$A - (a10^n + b10^{n-1})^2$	Middle: dividend
$2a\ .10^{2n-1}$	$2(a\ .10^n.10^{n-2} + b\ .10^{n-1}\ .10^{n-2})$	Lower: divisor

Several remarks will be essential for us here. The algorithm derived from the square root extraction, when one deletes the first part of the process, in fact solves a quadratic equation. For example, if we consider the shortened algorithm starting at step 8, it yields a root of the equation

$$x^2 + 2a\ .10^n\ .x = A - (a\ .10^n)^2.$$

If we look at the terms present in the array of numbers on the surface, whether we are at step 8 or step 16, in the middle row we have a so-called dividend, and in the lower row a so-called divisor or fixed divisor.

These terms (to which modern terminology refers, respectively, as the constant term and the coefficient of x in the quadratic equation) are precisely the two operands of the operation-quadratic equation as described in *The Nine Chapters*. In other words, in both contexts, not only do the operation-equations have only two operands (what in the modern concept of quadratic equation is the coefficient of x^2 is not identified as a term of the operation), but they also share the same two operands. Moreover, these two operands bear cognate names in both cases. It is unlikely that the correlation is fortuitous, but more is involved than these two features. It is by a kind of root extraction, that is, an algorithm derived from an execution of root extraction, that the quadratic equations read in some temporary configurations of this process are solved. This discovery echoes the prescription of the solution of the operation-equation in problem 19. Finally, in the proof of the correctness of root extraction, these quadratic equations correspond to the figure of a gnomon (see figure 14.8). If one extends the gnomon, one transforms it into a rectangle similar to that obtained by Liu Hui in his commentary on problem 19. Interestingly enough, in the technical terminology at that time, gnomons were designated by a term *ju* 矩, which was also a general term for a rectangle.[19]

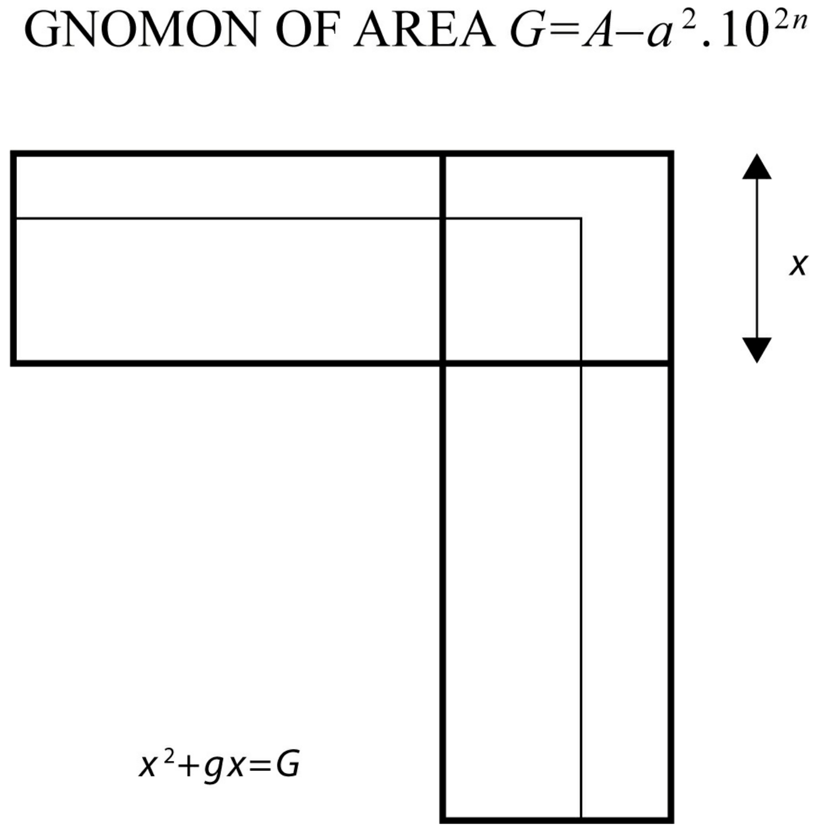


FIGURE 14.8. The gnomon or the rectangle as the figure of the quadratic equation

All these elements show that we have a complete correlation between, on the one hand, the operation-equation solving problem 19 and, on the other hand, the equation as it appears within the process of root extraction and as it can be detached from it thanks to the geometrical interpretation carried out in the context of the proof. That this relation is meaningful in the eyes of the authors of *The Nine Chapters* is clearly indicated by the prescription of the operation-equation in the procedure following problem 19 as a square root extraction.

Bringing the operation-equation of problem 19 in relation to these specific moments in the process of root extraction solves many questions I have not dealt with so far. First, the connection established between root extraction and the solution to problem 19 suggests a hypothesis for interpreting the designation of one operand of the operation-equation as a joined divisor. Indeed, in step 15 of the reconstructed process of root extraction (table 14.2), when the number in the auxiliary row below, $b.(10^{n-1})^2$, is deleted by being added to the row above it, that of the divisor, the text of the algorithm reads: "[Step 15] What has been obtained in auxiliary *joins* the fixed divisor." The commentator interprets the value computed by this operation of "joining" in the position of the divisor as representing the length of two tinted rectangles. On the one hand, it corresponds to the term in x in the equation thus constituted within the process of root extraction. On the other hand, in the correlation between root extraction and problem 19, this value corresponds precisely to what is designated as a joined divisor.[20] This fact illustrates how precise the correlation between the two contexts is. The remark even suggests that we are presented with more than a correlation here. The terminology seems to state how the operation "quadratic equation" was created on the basis of the process of root extraction. I return to this point below.

Further, we now understand how the operation-equation is solved. The execution of the operation-equation is indicated by the names of the operands and the prescription "one divides … by extraction of the square root." This practice reminds us of how the text for root extraction was formulated with the terminology of division. The terms invite the reader to rely, in a specific way, on the algorithm given for another operation.

We also understand why Liu Hui establishes the operation-equation as he does, and how his commentary relates to the text of *The Nine Chapters* for problem 19. The connection lies in the algorithm of root extraction and the proof of its correctness, in which the two-term equation is placed in relation to the figure of the gnomon-rectangle.

These hints confirm, at the same time as they outline, the intimate relationship between the quadratic equation and the algorithm of root extraction as they occur in *The Nine Chapters*. Most important, these conclusions reveal a second facet of the concept of equation. In this framework, the equation appears to be an operation with two operands. The beginning of the second section reached this conclusion by observing the structure of the text of the procedure in *The Nine Chapters*. Now, this conclusion is confirmed through observing the process of root extraction and the procedure extracted from it to execute the operation-equation. This confirmation brings a new piece of information: we see how the two operands were probably noted on the surface. Equations fundamentally consisted of two positions, which illustrates our previous remarks about positions as key components of operations in the culture under study. I refer to such notation as "written diagrams." This fact allows me to clarify a point. Despite their close links, root extraction and quadratic equation remained distinct operations: this appears clearly when we observe that the former operation had a single operand whereas the latter had two. Similarly, the analogy shaped between division and root extraction did not mean that the two had become the same operation.

Let me summarize what I have said about this facet of the equation in *The Nine Chapters*. Its two operands are those remaining on the surface for computing at some points of the algorithm for root extraction. The execution of the operation derives from root extraction, by removing an initial part in the latter algorithm. The prescription "one divides … by extraction of the square root," which concludes the procedure for problem 19, indicates that the resolution of the equation was carried out by part of the algorithm of root extraction.[21]

In fact, the concepts of algebraic equation to which Chinese sources attest until the sixteenth century appear to have all been perceived as depending in the same way on root extraction. This feature characterizes a subfamily, in the larger family of concepts of equations, in which we find concepts specific to the tradition considered in this chapter. In the subsequent centuries algorithms for root extraction would undergo changes. Accordingly, the concept of equation would also experience transformations. However, throughout the centuries, the concept and treatment of equations maintain their essential link to root extraction.

The connection established between the operation-equation and the truncated square root algorithm solves another problem about the first facet of the equation (its graphical inscription as a rectangle): why the commentary on problem 19 spoke of the sum of the areas that were originally north and south of the square town as forming "the area outside the corner." The analysis above has shown that the quadratic equation deriving from square root extraction is attached to the figure of a gnomon (see figure 14.8). This gnomon of area G is composed of a square with side x and two rectangles whose total area is gx. The figure of the gnomon shows why these two rectangles can be considered as forming "the area outside the corner." In effect, the commentators use the term "corner" to designate the successive squares in the corner of the square A, whose side represents the still unknown part of the root. The fact that Liu Hui, in his commentary on the procedure for problem 19, speaks of "the area outside the corner" to designate the graphical element corresponding to the term in x makes sense in relation to his own commentary on square root extraction. Further, this indicates that Liu Hui was writing the former commentary with reference to the latter. He also related the first facet of operation-equation to root extraction.

In conclusion, every question about the solution to problem 19 or its commentary finds its explanation in the algorithm of root extraction or in the commentary Liu Hui composed on it. These various elements confirm that the quadratic equation as conceived in our first time period in China was an operation depending on square root extraction and attached to a geometrical figure deriving from the proof of its correctness. These characteristic features are so removed from those we associate with quadratic equations that some analysts fail to recognize a family resemblance. In my view it is beyond dispute. In addition, a conceptual history of science can be global only if we learn to perceive such family resemblances efficiently and appreciate the variety of forms present-day concepts have had in all conceivable pasts.

The double face of the equation explains why the two meanings ("dividend" and "area") of the term *shi* 實, which designates the "constant term" G of the equation, are both active. When the first facet of the equation is considered, the meaning of "area" refers to its geometrical figure. Its use is correlated with that of terms like "corner" to designate pieces in this extension. By contrast, for the second facet, the meaning of "dividend" comes to the fore, according with the perception of the other operand as a joined divisor. The link between concepts of equation and the figure of the gnomon will be essential in the subsequent time period.

The Connection between the Interpretation of Equations and the Description of a Culture

We have seen that the procedure for solving problem 19 brought into play an operation with two operands, the "dividend" and the "joined divisor." These terms could be interpreted as corresponding to positions remaining on the surface at specific moments of root extractions. This explained why the solution of the operation-equation was prescribed as an extraction. We have also seen that, in Liu Hui's commentary on root extraction, these two terms corresponded to elements in a gnomon. This gnomon/rectangle is precisely the geometrical figure through which Liu Hui establishes the correctness of the equation solving problem 19. The connection between procedure and commentary for this problem is exactly the same as the connection between the algorithm and its commentary at the corresponding moments of the process of square root extraction. The connection links the equation seen as an operation (first facet) and the equation written as a geometrical figure, that is, as a rectangle, opposing a "corner" to a "joined divisor" (second facet). In that way, the discussion shed light on the concept of equation at that time in China.

We can now revisit briefly how our interpretation of the final operation of the procedure given after problem 19 relied on our knowledge of the mathematical culture in the context of which these texts were written. While examining the method used, I emphasize how the concept of equation thus revealed adheres to a given way of practicing mathematics.

My interpretation of the procedure following problem 19 relied on specific features of the practice of writing down texts for algorithms. We have seen how in the sources examined, texts for algorithms made a strategic use of referring to other algorithms. This makes sense in a context where the relationships between operations are meaningful. Interpreting these texts requires that these references be taken into account. Moreover, we observed names designating key entities of an operation. Their design contributed to shaping relationships between algorithms.

These specificities in the use of terms are correlated with the fact that texts of algorithms refer to the execution of procedures on a surface on which numbers were represented in an ephemeral way and placed in "positions." Physical properties of the tool for computation, and the practice with this tool, leave their mark in specificities of the texts for algorithms recorded in writings. Most of what we know about the practice with this tool has to be recovered through clues gleaned from the texts. Restoring this practice, and in particular the process of root extraction on the surface, yielded essential information on how "quadratic equation" derived from a temporary configuration in this process. This allowed us to establish its mode of solution. It also shed light on its operands. First, the relation between the algorithm for root extraction and the operation-equation, as evidenced in *The Nine Chapters*, suggests reasons the equation in that case has only two operands. Second, it shows that operands correspond to positions, on which the procedure of solution operates: the dividend in the middle line and the joined divisor in the line under it. This kind of notation, which was not included in the pages of writings in the first time period, made its appearance in texts of the second time period. This facet of the concept of equation under consideration captures how the concept adheres to the practice with the surface used for computations. The concept of operation-equation and its solution stem from another algorithm, more precisely, from the layout and execution of this algorithm. This concept of quadratic equation derives its nature of being an operation from the surface.

We had to take into account two elementary practices of the mathematical culture under study to interpret the text: the practice of proving the correctness of algorithms and the practice with diagrams connected to it. To interpret Liu Hui's commentary on the final operation of the procedure solving problem 19, I brought into play how the practice of proof relied on diagrams to formulate an "intention" or a "meaning" for elementary operations or subprocedures in a procedure. Taking these practices into account disclosed two key facts. First, Liu Hui's interpretation of some computations within the process of root extraction (steps 7–8 and steps 15–16) introduced the gnomon as the figure with which one could make sense of the following computations in the algorithm. This figure apparently played a key part in detaching the quadratic equation from the process of root extraction and giving it autonomy. Second, conversely, this figure, with the related figure of the rectangle, might have provided tools with which equations could be established. This is how we interpreted Liu Hui's commentary on the equation for solving problem 19. However, as I have stressed, this fact holds much more generally, providing us with a key to understanding all the equations established by the third-century commentators Liu Hui and Zhao Shuang. Restoring the "intention" as formulated in the proof, and the diagram supporting it, thus helps us capture another key facet of equation in that tradition of ancient China: the equation as a standard statement. In this context, the statement is written diagrammatically, in the form of a rectangle opposing a square of unknown side and a rectangle attached to it (these pieces correspond to the colors in Liu Hui's diagram; see figure 14.7). It is extracted from the diagram and the proof in exactly the same way as the array and the procedure solving the equation were extracted from the layout and the algorithm. Moreover, both are taken out of root extraction at exactly the same point of the process.[22] Like the first facet, and for the same reasons, this second facet of the equation leaves no graphical trace in the writings before the eleventh century, yet it clearly informed the earliest sources we have.

To conclude, we can see how the interpretation I developed through taking elementary practices into account establishes the nature of the concept of equation in the tradition of ancient China under examination. I have shown that a quadratic equation had two key facets, neither of which surfaced in the sources except indirectly. However, these facets are essential to take into account, not only to grasp the nature of and work with equations at that time in China but also to understand the changes in the concept of and work with equations in the long term. Further, each of these facets adheres to distinct features of mathematical practice in relation to which our sources were written down. The surface on which computations were executed yields the equation as operation and provides the means to solve it. The interpretations in proofs of the correctness of algorithms and the diagrams on which they are based yield the equation as statement and the means to establish it. These remarks summarize in which respect we can correlate features of the mathematical culture and the concept of equation. Note that each of the two facets of the equation indicates the limitations of the concept in China at that time. The operation bore only on two operands, both of which were "positive" in modern terms. Only later will other types of quadratic equations be accommodated within this framework. This feature of the concept of equation in the first century further reinforces the conclusion of the dependence of the operation-equation on root extraction.

We now have an adequate basis to approach the concept of quadratic equation in our second time period, which seems to have begun in the eleventh century.[23]

A Subsequent Culture and a Subsequent Concept of Equation: Continuities and Differences

I take as a record of the second time period the last part of the final chapter of Yang Hui's *Quick Methods*. In fact, there Yang quotes Liu Yi's *Discussing the Source*, which probably dates from the eleventh century. It is not clear where the quotation starts, where it ends, or how exactly Yang adds to it—this is not relevant here. My only aim is to point out a new way of working with quadratic equations and a new concept of equation in the context of and in relation to another mathematical culture.[24]

As was mentioned above, the writings from the second time period are in sharp contrast with those from the first time period. The most conspicuous change is that practices of working with diagrams and with a surface for computing now leave traces in the writings as illustrations (see figure 14.2). These traces on paper attest to practices in continuity with earlier practices outlined above, but they also exhibit transformations. The transformations I use to illustrate this point are those that can be perceived in practices with diagrammatic elements. My claim is that, despite a strong continuity, by the eleventh century diagrams play a new and more fundamental role in the treatment of equations. It is essential to observe the change in the elementary practice with diagrams to interpret them adequately. As for equations, the sources testify to the fact that they are worked out within a conceptual and material framework similar to that described for the first time period. However, within this framework, the concept of equation and the algorithms solving equations show major extensions. My aim now is to describe the changes in the practice, the changes in the concepts, and the relation between the two.

To begin with, let us observe how Liu Yi's *Discussing the Source* testifies to how one worked with equations at the time. The first problem in the quotation of Liu Yi's book, problem 43 in the final chapter of Yang's *Quick Methods*, is the best introduction to this question, since this problem sets a frame of reference for the following problems. Its statement reads as follows: "The area of a rectangular field is 864 *bu*. One only says that the width fails to be equal to (that is, is less than) the length by 12 *bu*. One asks how many *bu* the width is" (Kodama 1966, 91).[25] The statement of the problem is immediately followed by the answer, "24 *bu*," and a procedure that reads: "One puts the area as dividend. One takes the *bu* (by which the width) fails to be equal to (the length) as what joins the square (or: what is appended to the square). One divides by this by extraction of the square root" (91). This procedure brings into play an operation-equation in a way similar in almost every point to the procedure solving problem 19 in *The Nine Chapters*. It describes how to derive a dividend and an operand called "what is appended to the square" (*congfang* 從方), which corresponds to the old term "joined divisor." I return to these terms below. The procedure is then concluded by the same formulaic expression. In modern terms, the procedure corresponds to the equation $x^2 + 12x = 864$. If we set aside the change in the topic of the problem and the change in the terminology, nothing seems to have changed since the first century. In this sense at least, we perceive the continuity with what was described above. As for the changes, they appear inessential. Yet, we would be mistaken to consider these two small changes minor. I first focus on the statement of the problem.

To understand what is at stake in the change of topic, it is helpful to read a quotation of Liu Yi's preface to his book, which Yang places before problem 43 and repeats in two other places in the same book. This statement clarifies the new status of the equation and its relation to the topic of the problem: "Sir Liu [Yi] from Zhongshan said in his preface: 'As for the procedures of mathematics, from any point we start engaging [with them], one ends up with [systematically the same mathematical entity of] the rectangular field/figure [直田]' " (Kodama 1966, 91). This statement probably held true for the whole book. Whatever the situation dealt with in a problem, the quotation asserts, its solution reduces it to a "rectangular figure" and then—one might probably add—to the procedure for determining its root. If we compare the declaration to what follows, we realize that, in this new context, "rectangular figure" is a term designating the concept of quadratic equation, for which no other technical term can be found in that text. If such is the case, this remark implies an essential fact: problem 43, like those following it and similar to it, does not deal with an arbitrary situation but with the "equation" itself, as Liu Yi and Yang understood it. In this sense, the topic of the problem has undergone an important change.

In Liu Yi's text, the procedure is not followed by an equation written in modern terms but by the diagram of a rectangle (see figure 14.2, first rectangle from the right). The rectangle illustrates the situation described in the statement of the problem. It can also be interpreted as a graphic formula writing down the equation as Liu Yi and Yang conceived of it in this context.[26] Here, more precisely, the diagram writes the operation-equation, whose operands have values determined by the procedure placed after the problem.[27] The diagram shows a square, having as its side the width, to which a rectangle is appended, one side of which is the unknown and the other equals 12 *bu*. The overall rectangle is the "rectangular figure" of the problem. The captions inscribed on the diagram make this point clear. They refer to terms introduced in the procedure, including operands of the operation-equation. The caption placed under the diagram confirms this reading, since it comments: "Above, this is one piece of the square with [side] the width. Below, this is one piece of what is appended to/joins the square of [side] the width." The way in which, as we shall see, such a graphic formula of the equation is put into play shows that it was used as a base in working with the operation-equation.

It is striking that the rectangle "writing" the equation is graphically identical to the first facet of the quadratic equation in the first time period. In this respect, there is some continuity in practice, despite a change in the materiality of the diagram. There was also continuity in the way a procedure introduces the operation-equation. The terminology for its operands looked quite similar. However, the caption to the diagram seems to testify to the fact that, compared with earlier sources, the word *cong* ("join") that occurs in the designation of the term in x in both contexts, is now to be interpreted in a new way. The divisor is no longer characterized through the fact that, geometrically, it lies outside of the corner. Rather, *cong* designates the corresponding geometrical element by the fact that it is "appended to" the square.[28] A reconceptualization of the situation can be grasped in the change of terminology. It may also be what is at stake in the replacement, in this context, of the term *ju* ("gnomon/rectangle") by the term *zhitian* ("rectangular figure") to designate the rectangle.

This reconceptualization of the shape and structure of the rectangle is reflected in the two following components in the text of problem 43. First, the graphic formula that the rectangle constitutes illustrates the reason that the algorithm yielding the root of the equation is correct (see figure 14.2, second figure from the right).[29] The caption makes clear this is the function of the second diagram in the author' eyes, since it reads: "Diagram of the detail of the procedure [showing] the values of the pieces in a root extraction having [something] appended to [the square]." In that diagram, the area of the global rectangle, and consequently of its components (square and rectangle), is cut into pieces to show the meaning of all computations in the algorithm, if interpreted graphically. This diagram attests to how the writing of the equation in the form of a rectangle provides a notation used to work with the equation. This remark holds true for the following problems. The captions connect the pieces in the diagram with what happens on the surface used for computations, during the process of root extraction. This process is illustrated by the subsequent diagram, which is of a different kind. It is composed of three subdiagrams showing three configurations of numbers at different moments in the computation, much in the same way as what we reconstructed for the algorithm of *The Nine Chapters* (see figure 14.2, left page). The captions outside the subdiagrams indicate the specific moment when the configuration is extracted. They also explain what the rows contain. Below this diagram, a continuous text describes the algorithm.

The continuity with the numerical treatment of the quadratic equation of the first century, as discussed above, is manifest. However, two key transformations can be captured in the configuration of numbers. They illustrate again my thesis in this chapter: restoring material practices is an essential task, which offers tools for a more precise conceptual history.

First, in contrast to the algorithm in *The Nine Chapters*, the lower line is constantly present in the scheme of computation. It corresponds to what for us is the term in x^2 in the equation. In correlation with its presence throughout the computation, subsequent problems deal with operations-equations for which the related coefficient is different from 1 or even negative. Accordingly, modifications of the algorithm for root extraction are described. This is the first extant piece of evidence among Chinese sources that a third operand is attached to the quadratic equation.

The second key transformation can be easily described by reference to the algorithm for root extraction in *The Nine Chapters*. In the first century the operation "quadratic equation" was extracted from root extraction, after the step when the square of the first digit is subtracted from area A (step 8; see table 14.2). The operation corresponds to the shape of the gnomon, and Liu Hui accounts for the correctness of the algorithm determining the successive digits of the root by showing that each phase amounts to taking a slice out of the area of the overall gnomon, which has the shape of a thinner gnomon and whose width corresponds to the next digit determined. More precisely, for each digit of the root, the algorithm determines a divisor, which corresponds to the length of the thinner gnomon (if it is stretched), and it suffices to multiply this length by the corresponding digit of the root to obtain the area to be subtracted from the overall gnomon. Liu Yi's algorithm differs from that algorithm precisely on this point. It dissociates, and places in two separate rows, the component from the first-century divisor that, in modern terms, corresponds to the term in x in the equation (in third-century terms, this is the component of the gnomon "outside the corner"), and the component deriving from the "corner." The former component is placed in the row corresponding to "what joins/is appended to the square," whereas the latter yields the "divisor of the square."

This fact has two important consequences. The dissociated lines in the configuration of numbers now correspond each to a component in the diagram accounting for the correctness of the algorithm. The "divisor of the square" corresponds to the upper part, whereas "what is appended to the square" relates to the lower part. We thus see here the second point in which the reconceptualization of the graphic formula of the equation described above finds an echo. Here the reconceptualization is reflected in the writing of the operation-equation on the surface for computations, or, to use the expression introduced above, the written diagram. The new text is crafted in such a way that the components of the written diagram and those of the graphic formula correspond to each other in a transparent way.

The second consequence of the separation of the divisor line into two relates to the new treatment of operations-equations. The component corresponding in our terms to the term in x is being dissociated from the others, in the written diagram with which the equation is inscribed on the surface for computations. Accordingly, new operations-equations are considered, in which this operand can be negative, and new algorithms are described to deal with the various possible signs of the operands of an equation.

I have now introduced the fundamental elements that, in the context of the mathematical culture to which Yang's text attests, are involved in the new treatment of the operation-equation. They include problems, texts for procedures, graphic formulas that diagram the equation, and written diagrams that illustrate the numerical computations. We thus see how the treatment of equation presented by Liu Yi is in continuity with earlier treatments. We could not have grasped this fact had we not restored the diagrams used in the first time period.

Further, besides continuity, the text testifies to conceptual transformations in the treatment of equation. Liu Yi's text illustrates four directions in which his operation-equation differs from the previous one. First, the rectangular figure is now stated to be a universal object. Second, his writing attests to a progressive identification of a third operand—the term in x^2—in the quadratic equation. Third, we can observe in his text how negative marks on operands of the operation-equation are introduced, in relation to the widening of the range of equations considered. Fourth, Liu Yi's text testifies to the transformation of the algorithm solving the equation in a way that brings to light a higher homogeneity in the changes that execute the computation of the root. I have mentioned hints that suggest a correlation between the new way of working with equations and the transformation of the operation-equation.

I cannot describe here each feature of the changes in the mathematical culture and the conceptual approach. I examine only how the graphic formula is used in a new way, to work on new questions with the operation-equation. It is all the more important to focus on that aspect since it requires uncommon practices of interpretation. I claim that these practices of interpretation of the diagrams bring to light meanings that would otherwise remain hidden.

I have mentioned that Liu Yi's text considers new types of quadratic equations. How do the graphic formulas write down these new equations? Let us examine some cases.[30] Problem 44 deals with the same rectangle as problem 43, but now the length L is required, while the area A and the difference $L - l$ are known. The procedure given by Liu Yi involves an operation-equation corresponding, in modern terms, to

$$x^2 - 12x = 864.$$

It is written graphically as such, as is shown in figure 14.9. The caption on the left component of the diagram expresses the fact that the rectangle corresponding to $12x$ is the piece lacking in the square x^2 and the reason that the remaining area is equal to 864. In other words, the diagram connects the statement of the problem and the operation-equation for solving it. Further, the diagram writes the equation, and writes it in the standard form, expressing how a dividend/area is yielded by operations between the terms involving the unknown (divisors). The graphic formula still brings together a square and two rectangles. However, the graphical connection has changed in relation to the fact that the diagram writes a new type of equation. This is the first manifestation of a phenomenon that permeates the whole text and takes various shapes (figures 14.9 and 14.10).

Problem 46 shows a variation of the pattern just described. Its data consist of the area A of a rectangular figure with sides L and l and the sum $L + l$; the unknown sought is the width l. In modern terms, the operation-equation for solving it can be written as

$$60x - x^2 = 864.$$

Figure 14.10 displays the graphic formula for writing the equation. Its overall rectangle has the unknown (l) and $L + l$ (which is equal to 60) as its sides. The diagram indicates how this overall rectangle combines the fundamental rectangle with area A and the square of the unknown, that is, the width. Its captions disclose interesting new elements, which are revealing of the part played by the graphic formula. The caption attached to the right designates the sum $L + l$ by the term "what is appended to/joins the square," namely, the lower square whose side is width l. This use reveals that the geometrical meaning of the expression had receded at that time and a functional meaning had come to the fore that related the geometrical element to the homonymous line, in the configuration of numbers on the surface for computations—the written diagram. This transformation in the meaning of the terms correlates with the changes in the concept of operation-equation.

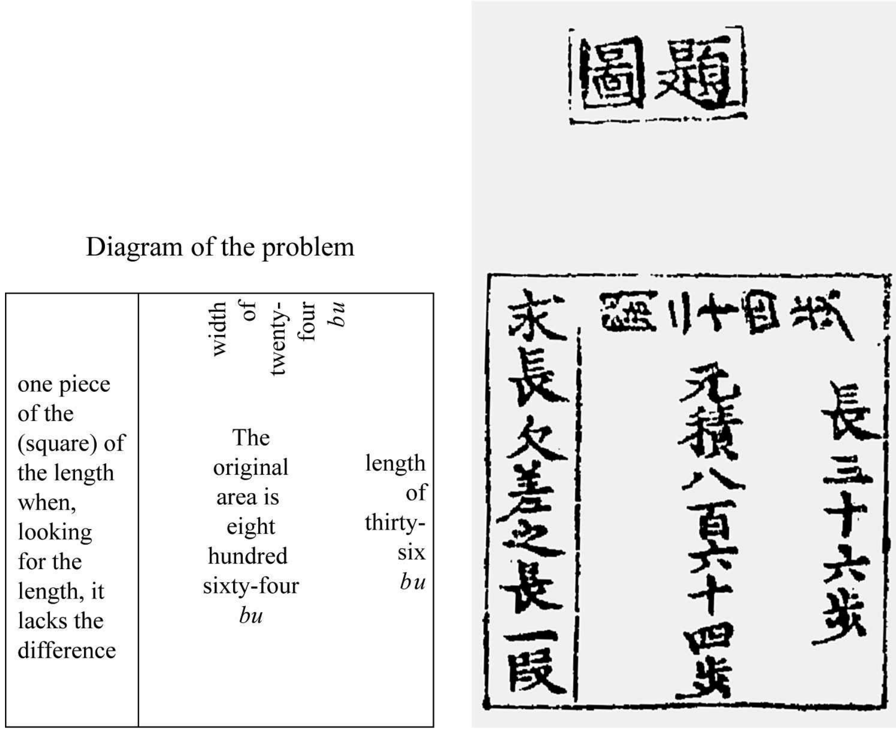


FIGURE 14.9. Liu Yi’s graphic formula for the equation corresponding to problem 44, in Yang Hui’s *Quick Methods for Multiplication and Division for the Surfaces of the Fields and Analogous Problems*, Korean edition from 1433, reprinted in (Kodama 1966, 92).

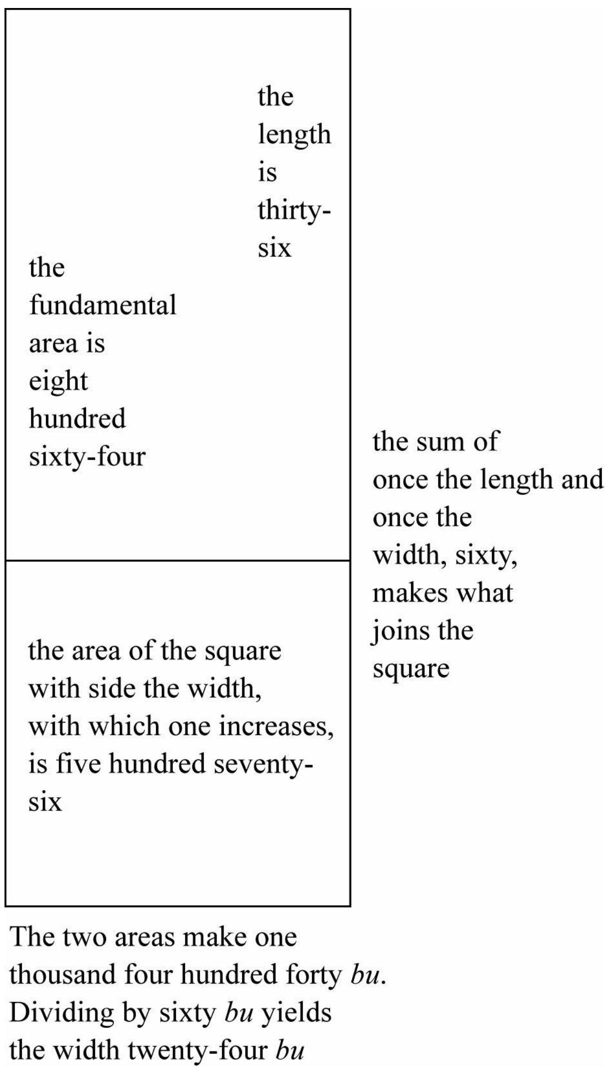


FIGURE 14.10. Translation of Liu Yi’s graphic formula for the equation corresponding to the first solution of problem 46 in *Quick Methods*.

In fact, the diagram combines two functions—it writes the equation. But it also serves as a base to consider the correctness of the algorithm given to solve similar operations-equations. Thus, the notation simultaneously serves two of the purposes that we pursue with symbolic notations. However, where we today would write a sequence of formal notations, Liu Yi brings together the various uses of the notation in a single diagram.

The general conclusion, important for my main topic, is that the graphic formula writes the equation in ways that must be described systematically if we want to interpret it adequately. We can project on it neither our expectation that the equation should be written discursively nor our belief that distinct facts should be addressed by distinct statements. The kinds of statement, as well as their use by the actors, need to be attended to if we aim for a conceptual history of equations in China.

The same equation will be written in another way, for problem 47, whose outline is identical to problem 46 but whose unknown is the length. The equation is thus still

$$60x - x^2 = 864.$$

The graphic formula brings into play a new ingredient. In line with the extension of the operation-equation announced above, the term "negative" occurs in the caption of the piece corresponding to the operand "square" (in modern terms, x^2). In the following problems, another graphical element is used to denote negative operands: colors. For instance, in problem 52, the operation-equation corresponding, in modern terms, to

$$A = 312x - 8x^2$$

is written graphically. The negative operand appears in black. The mathematical culture observed in the first time period employed colors to mark diagrams, as described above, and this technique is taken up in the subsequent context. This fact displays a form of continuity between the two time periods in the practice with diagrams. However, the practice is now invested with new meanings, in relation to a major conceptual change. Equations can have negative operands, and color is used to denote them. This example illustrates how ancient features can be put into play to extend the possibilities of expression of the graphic formula. It is essential to interpret color if we want to understand how these graphic formulas fulfill two of their essential functions.

One of these functions, as we have seen, is to establish the correctness of the various algorithms put into play to find out roots, in relation to the nature of the equation solved. In this context, color is used to make a graphic formula display the key point of a proof, as illustrated by the diagram associated with the second algorithm given for problem 46.[31] The second function of the graphic formula in which color will be used is that of providing support for establishing the equation. For problem 46, the equation established derives, in the process of proof, from the equation solved. Color is also used at the beginning of the solution of a problem to establish how it can be solved using an equation.

In conclusion, if we compare this practice of diagrams with that for the first time period, we see that, although the same graphic formula of the rectangle is used, it is used in a new way, in conformity with the extension of the range of equations considered and the related change in their nature. Liu Yi's graphic formulas differ from the earlier samples, and they do more work than was previously the case. Some of the features of this new usage have ancient roots, such as colors. This continuity should not, however, hide the new meaning inscribed with these old techniques in Liu Yi's time.

Conclusion

We can now return to the theses expounded at the beginning of the chapter and see how the conceptual history of "quadratic equation" in China between the first and the eleventh century supports them. The case study I developed here shows the utility of describing how practitioners worked with various elements in their mathematical activity—here mainly problems, algorithms, the surface for computing, and diagrams—and how they connected these elements with one another. For such similar complexes of practices I suggest using the term "scholarly" or, here more specifically, "mathematical" cultures. I have argued that describing the mathematical culture in the context of which practitioners operated provides essential tools to interpret sources. This point is more enhanced in a case where sources are a challenge for the interpreter. In the case described here, this method helped me determine the nature of the concept of equation in the first century, that is, an operation-equation, and the transformations of this operation-equation in the succeeding centuries.

The descriptions of different mathematical cultures allow us to grasp continuities as well as transformations in ways of doing mathematics in China between the first and the second time period. In each of these two contexts, the concept of equation correlates with ways of working with diagrams and the tools for computations. Such an approach highlights material dimensions of conceptual history.

Despite differences, strong continuities, both material and conceptual, between the two ways of conceiving equations can be recognized. These continuities define a tradition of working with equations as operations that, to my knowledge, cannot be identified in sources other than Chinese ones. However, the existence of this tradition does not imply any kind of determinism, according to which the concept of equation, once set in a framework, could only develop within the bounded space of this framework. This fact is illustrated quite strikingly by the later history of concepts of equations in China. By the thirteenth century, Chinese sources attest to another concept of equation that presents strong continuities with earlier concepts but redefines a tradition in an entirely new way. Li Ye's *Sea Mirror of the Circle Measurements*, completed in 1248, illustrates this phenomenon. In this book, equations are noted as written diagrams, in continuity with the way the operation was inscribed on the surface on which computations were carried out. Only one of the two facets of earlier concepts of equation survives (Li 1958, chaps. 22, 23).[32] The graphic formula recedes in the background and is replaced by other (algebraic) means of establishing the operation-equation that also derive from the numerical facet of the equation. The establishment of this new way of responding to inherited tradition goes together with new developments in the understanding and treatment of equations.[33]

All the concepts of equation that can be identified through Chinese sources, however, share a common feature: they consider the equation as a numerical operation. In this chapter I have shown how this feature adheres to the practice with the surface on which to carry out computations. Despite this adherence to a stable feature of the cultures within which equations were used with in China, this approach to equations is not so typically "Chinese" that it could not circulate. In fact, a numerical approach to the solution of equations quite similar to that one, both materially and mathematically, suddenly occurs in Arabic sources in the twelfth century. In *On Equations*, by Sharaf al-Din al-Tusi,[34] the concepts and treatment of equations combine features of equations coming from the tradition established by al-Khwarizmi (first half of the ninth century) and that illustrated by Omar Khayyam (1048–1131). They further incorporate a new way of approaching the solution of equation, which presents striking similarities with the approach that had developed in China. So far, no historical evidence has been found that this was due to circulation, and yet I believe it is highly probable.[35] Whatever the case, *On Equations* testifies to the possibility of merging different concepts and treatments of equations into a single whole, thereby demonstrating that the concepts and modes of solution shaped in China could be adopted in other contexts and interact with other approaches. This conclusion suffices to establish that even though concepts and results may adhere to features of scholarly cultures, they are not condemned to remain within these boundaries and be incomprehensible for other scholarly cultures.

Notes

I am grateful to Evelyn Fox Keller and Bruno Belhoste for their comments on an earlier version of this chapter. I also thank the participants in the workshop at Les Treilles in June 2011 for their remarks, in particular my three commentators, Emmylou Haffner, Donald MacKenzie, and David Rabouin. The research presented in this chapter is part of the work that led to the European Research Council project SAW (ERC grant agreement 269804). Many thanks to Karen Margolis for sharing her thoughts with me about the formulation of this chapter. The chapter was completed while I was in Seoul, benefiting from the hospitality of the Templeton Science and Religion in East Asia project hosted by Science Culture Research Center, Seoul National University.

1. In this chapter, I take this topic only as an illustration, outlining the argument without giving any detail. The argument draws on several publications, which constitutes the core of a book I plan to write.

2. The reader can find a detailed treatment of the issue of how we can describe cultures using scientific documents in Chemla 2010a. Chemla 2009 presents an outline of the argument.

3. The symmetrical problem would be to introduce a new term for each distinct concept. Usually such historical practice is carried out in an asymmetrical fashion. This is how we end up with the idea that there was nothing in China, no philosophy, no mathematics, and so on.

4. Cullen 1996 offers a translation of *The Gnomon of the Zhou*. The commentaries still await systematic study. Chemla and Guo 2004 contains a critical edition and a translation into French of *The Nine Chapters* and Liu Hui's commentary. In the present chapter, I rely on this critical edition.

5. We can establish that, as for the number system commonly used today, the basis for the number system was 10. Moreover, a digit derived its meaning from the position in which it was put, in the same way as, when we write 123, 1 derives its meaning of "a hundred" from its position in the sequence of digits.

6. The following statements require qualification, but I must skip details (see Chemla 2003, 2009, 2010a, 2010b, and Chemla and Guo 2004).

7. For the moment, we lack evidence to date this change. It must have occurred between the eighth and the eleventh century.

8. Lam 1977 contains a full translation and discussion of *Quick Methods for Multiplication and Division for the Surfaces of the Fields and Analogous Problems*. Guo Xihan 1996 constitutes a guide to its reading. Te 1990 and Horiuchi 2000 discuss the remaining evidence on Liu Yi and in particular his treatment of quadratic equations. Here I omit scholarly discussion on matters of date and attribution, concentrating instead on the concept of equation to which this writing bears witness and its relation to a mathematical culture.

9. I have begun to describe this shift (Chemla 2001), but further research is required.

10. All the evidence we have for the first time period shows equations having the same features as those established in this part of the chapter.

11. I use bold characters for terms on which I shall comment below. See Chemla and Guo 2004, 689–693, 732–735 for the Chinese text, its translation, and its interpretation.

12. To carry out the algorithm, other functions are introduced and terms are attached to them; see below.

13. See Li and Du 1963, 64–67, 1987, 53–55; Qian 1981, 51; Li 1990, 112–114, 404–405, 1998, 728–729; Martzloff 1997, 228–229; and Shen et al. 1999, 212–213, 507–512. I return to these authors' interpretation below.

14. I stress here that at that time the "equation" has two operands and not three. This will allow us to perceive a key change in the concept of equation later. Other historians have not noticed this change.

15. In my talk at the Stanford University–REHSEIS (Recherches en Epistémologie et Histoire des Sciences et des Institutions Scientifiques) Workshop on diagrams, organized by S. Feferman, M. Panza, and R. Netz (October 2008), to designate Liu Yi's diagrams for equations (see the third section), I borrowed the expression "graphic formulas" (*gezeichnete Formeln*) from Hilbert 1900. I also

borrowed from Hilbert 1900 the expression "written diagrams" (*geschriebene Figuren*), which I use below.

16. A critical edition, an annotated translation, and references to other publications on the topic are given in Chemla and Guo 2004, 322–329, 362–369.

17. The adjustment of the value of the divisor is made possible through computations carried out in a row that the scheme of root extraction places under the fundamental three-row scheme of division. At the time of *The Nine Chapters*, this row below was considered auxiliary.

18. On the use and meaning of colors as well as unit-squared paper to make diagrams in ancient China, see Chemla 1994, 2001, and 2010b.

19. See the glossary in Chemla and Guo 2004, 943.

20. The verb "to join" occurs at the same place in the two contexts and correlates the joined divisor with the divisor of a root extraction, at the moment when it has been joined by the number in the row under it.

21. Li and Du 1963, 61–66, also interpret the quadratic equation and its solution as, respectively, a temporary configuration on the surface for computing in a root extraction and the part of the root extraction starting at this point. For these authors, too, quadratic equation derives from root extraction. However, my interpretations differ in several aspects. First, I do not restore the algorithm for root extraction in the same way. As a consequence, for Li and Du, the equation as it appears as a temporary configuration has three operands, not two—it includes a term in x^2. This interpretation does not allow them to see the transformation of the concept of equation to which later sources attest. Li Yan and Du Shiran do not refer to Liu Hui's commentary in their interpretation of the equation. They read the equation from the process of root extraction. All these features also characterize Jean Claude Martzloff's (1997, 224–229) account for algebraic equations. As a result, Li and Du's geometrical interpretation of the process of solution, as well as the concrete numerical process they restore, is similar to that in eleventh-century sources. Moreover, they do not emphasize the geometrical facet of equation in our first time period. They adopt another view on the connection between geometry and this equation. This leads them to read some geometrical problems and algorithms as amounting to solutions of quadratic equations by radicals (Li and Du 1963, 73–76). This interpretation seems contrived. Lastly, they interpret the term "joined divisor" as "following the divisor." In their view, this refers to the fact that this divisor "has a nature comparable to that of" (Li and Du 1963, 74) the other divisor in a root extraction. Revealingly, they refer to this other divisor by the term "square divisor," which in fact surfaces only in later sources. The translation into English in Li and Du 1987, 52–55, 61–63, does not convey the meaning of the original. Qian 1981, 47–51, 58–60, offers exactly the same interpretation, which dates back to the 1950s at the latest.

22. Li Jimin (1990, 112–114) does not interpret the equation in *The Nine Chapters* as I do. For him, geometrically as well as algorithmically, the equation is a square root extraction to which an auxiliary term was "appended": a rectangle is appended to the square to write down the equation; a line is appended to a line in the root extraction to record the term in x. This is how he interprets the term *cong* that I translate as "joined." I agree that this is how the equation would be understood in the eleventh century (see below). However, in my eyes, his view mostly anachronistically projects the concept of equation that characterizes the second time period onto the first time period. The same conclusion applies to Shen 1997, 288–289, 682–683, and Shen et al. 1999, 212. Consequently, Li cannot explain why in the first century the equation has only two operands, and in fact he seems not to have grasped the importance of this feature for the later history of equations. It seems to me difficult to understand why the initial term in x would be qualified as "appended," since, when it is placed on the surface to compute, there is no other divisor to which it could be appended. Revealingly, Li has to add the following sentence at the beginning of the algorithm for root extraction: "The appended divisor makes the fixed divisor" (113). For him the algorithm solving the equation is an extension of the algorithm for root extraction and not, as I believe, a procedure deriving from it. This interpretation implies that *The Nine Chapters* does not describe how the square root extraction is modified, nor does the commentary account for the correctness of the extended algorithm. Lastly, Li does not see the general importance of the shape of the gnomon in that early phase of the history of quadratic equation. Accordingly, he does not seem to grant any part in this context to the practice of proof. The same features hold true in Li Jimin's (1998) translation of *The Nine Chapters* into modern Chinese. In the same vein as Li and Du (1963), Li (1990, 367–368) appears to offer a contrived interpretation of quadratic equation.

23. Annick Horiuchi seems to have missed the second facet of the equation in the first time period (see Horiuchi 2000, 243), and hence did not grasp the continuity in all its dimensions.

24. Since 2007 I have been preparing a critical edition and translation of this text, which I refer to as Liu Yi's writing. I refer to the edition of Yang Hui's *Quick Methods* reprinted in Kodama 1966, 91–97, on the basis of the 1433 Korean reprint. Lam 1977, 112–133, contains a translation of the passage dealt with here. Note that in Lam 1977 the diagrams are not translated faithfully. Guo 1996, 229–279, provides elements for a critical edition and explanations.

25. The unit *bu* is used for measuring lengths as well as areas. For areas, the *bu* is a square unit having a side of 1 *bu*. The term translated as "field" acquired the more general meaning of "figure" in the third century at the latest.

26. Most historians have dealt only with the numerical facet of the concept of equation in Liu Yi's text, leaving aside the diagrams as if they were mere illustrations. We can identify two forms of anachronism in this interpretation. First, historians have projected onto the source our perception of the part played by figures in mathematics. This remark shows why recovery of the practice with diagrams is important. Second, they have read these texts through the lenses of the treatment of equations in China in the thirteenth century, as illustrated by Li Ye's *Sea Mirror of the Circle Measurements* (1248), within the context of the so-called procedure of the celestial origin. In that other context, diagrams play an entirely different role. This holds true for Li 1958, 185–188; Qian 1981, 154–157, 1966, 44–

47; and Martzloff 1997, 142.

27. Horiuchi (2000) revealed the universal character of the rectangular figure and its meaning, showing the part played by the rectangle as a tool with which to work on equations. I shall explain below the two main uses of the rectangle as a support for operations that she first discussed. However, my interpretation differs from hers in that I suggest that the rectangle writes the equation and that the reader had to interpret it as such. This interpretation derives from the fact that I take scholarly cultures in their variety into account. In my view, a substantial part of Yang Hui's treatment of the equation has to be read from the diagrams and has no counterpart in the discourse. This holds true for later texts that record similar treatments. Horiuchi felt "compelled" to recognize that the text contained only a diagrammatic treatment (251), while expecting a discursive treatment. Such a conclusion calls for a critical edition of diagrams, which I am currently preparing.

28. Li Jimin and other historians have projected this reading of the equation onto texts of the first time period (see notes 21 and 22 above for exact references). I believe they have missed a subtle change in the understanding of the operation-equation.

29. In this context, operations-equations were believed to have only one root. I deal with this issue in my book in preparation.

30. The transmission of the diagrams was problematic. For what follows, I am relying here on my work on the critical edition of the text in preparation.

31. Lam 1977, 260–262, explains the algorithm, but the figure given there does not correspond to that contained in the sources. For details, see Chemla forthcoming.

32. Since previous historiography of mathematics in China has mainly emphasized the facet of the equation represented by the written diagrams, overlooking the graphic formula, the break has appeared less dramatic than it actually is. The graphic formula has offered the support to establish the equation and address the correctness of the algorithm solving it. These two activities have also been for the most part overlooked.

33. Using terms introduced in chapter 1 in this volume, the cultural history of equations in China shows two distinct forms of "path dependence."

34. Rashed 1986 provides a critical edition, a translation, and an analysis of the book. Chemla 1992 is an outline of the following argument.

35. Adolf Pavlovitch Juschkewitsch ([1961] 1964) also believed it. He was, however, relying only on the source material available at the time. The discovery of *On Equations* by Tusi does not undermine the conclusion. As Donald MacKenzie noted during the 2011 discussion at Les Treilles, this chapter "draws an analytical distinction between cultures and concepts, when in the Geertzian notion concepts are surely at the heart of culture" (see Geertz 1973). The transformations and circulations this paragraph evokes show how this distinction might help us not to "orientalize."

References

Chemla, K. 1992. "De la synthèse comme moment dans l'histoire des mathématiques." *Diogène* 160: 97–114.

Chemla, K. 1994. "De la signification mathématique de marqueurs de couleurs dans le commentaire de Liu Hui." *Cahiers de linguistique—Asie Orientale* 23: 61–76.

Chemla, K. 2001. "Variété des modes d'utilisation des *tu* dans les textes mathématiques des Song et des Yuan." Preprint given at the conference "From Image to Action: The Function of Tu-Representations in East Asian Intellectual Culture," Paris, September 3–5. http://halshs.ccsd.cnrs.fr/halshs-00000103/.

Chemla, K. 2003. "Generality above Abstraction: The General Expressed in Terms of the Paradigmatic in Mathematics in Ancient China." *Science in Context* 16(3): 413–458.

Chemla, K. 2009. "Mathématiques et culture: Une approche appuyée sur les sources chinoises les plus anciennes." In *La mathématique. 1. Les lieux et les temps*, ed. C. Bartocci and P. Odifreddi, 103–152. Paris: CNRS.

Chemla, K. 2010a. "從古代中國數學的觀點探討知識論文化" ("An Approach to Epistemological Cultures from the Vantage Point of Some Mathematics of Ancient China"). In 中國史新論．科技史分冊 ：科技與中國社會 (*New Views on Chinese History: Volume on the History of Science and Technology: Science, Technology, and Chinese Society*), ed. P. Chu, 祝平一 , 181–270. Taipei: Lianjing 聯經.

Chemla, K. 2010b. "Changes and Continuities in the Use of Diagrams *Tu* in Chinese Mathematical Writings (3rd Century–14th Century) [I]." *East Asian Science, Technology and Society* 4: 303–326.

Chemla, K. Forthcoming. "How Did One, and How Could One, Approach the Diversity of Mathematical Cultures?" *Proceedings of the 7th European Congress of Mathematics*, Berlin, August 18–22, 2016, ed. Martin Skutella.

Chemla, K., and Guo, S. 2004. *Les neuf chapitres: Le classique mathématique de la Chine ancienne et ses commentaires*. Paris: Dunod.

Cullen, C. 1996. *Astronomy and Mathematics in Ancient China: The Zhou Bi Suan Jing*. Cambridge: Cambridge University Press.

Geertz, C. 1973. *The Interpretation of Cultures*. New York: Basic Books.

Guo, X. 郭熙漢. 1996. "楊輝算法"導讀 (*Guide to the Reading of "Mathematical Books by Yang Hui"*). Hankou: Hubei sheng jiaoyu chubanshe.

Hilbert, D. 1900. "Mathematische Probleme." *Göttinger Nachrichten* 3: 295; trans. M. W. Newson in *Bulletin of the American*

Mathematical Society 8: 437–479, 1902.
Horiuchi, A. 2000. "La notion de *yanduan*: Quelques réflexions sur les méthodes 'algébriques' de résolution de problèmes en Chine aux Xe et XIe siècles." *Oriens-Occidens* 3: 235–258.
Juschkewitsch [Youschkevitch], A. P. [1961] 1964. *Geschichte der Mathematik im Mittelalter*. Leipzig: Teubner.
Kodama, A. 児玉明人. 1966. 十五世紀の朝鮮刊—銅活字版數學書 (*Mathematical Books Printed in Korea in the Fifteenth Century with Copper Moveable Types*). Tokyo: 無有奇奄雙私刊.
Lam, L. Y. 1969. "On the Existing Fragments of Yang Hui's *Hsiang Chieh Suan Fa*." *Archive for History of Exact Sciences* 6(1): 82–88.
Lam, L. Y. 1977. *A Critical Study of the Yang Hui Suan Fa*. Singapore: Singapore University Press.
Li, J. 李繼閔. 1990. 東方數學典籍 ——— 《九章算術》及其劉徽注研究 (*Research on the Oriental Mathematical Classic "The Nine Chapters on Mathematical Procedures" and on Its Commentary by Liu Hui*). Xi'an: Shaanxi renmin jiaoyu chubanshe.
Li, J. 李繼閔. 1998. 九章算術導讀與譯註 (*Guidebook and Annotated Translation of "The Nine Chapters on Mathematical Procedures"*). Xi'an: Shaanxi renmin jiaoyu chubanshe.
Li, Yan 李儼. 1958. 中國數學大綱 (*An Outline of Chinese Mathematics*), rev. ed. 2 vols. Beijing: Science Press.
Li, Yan 李儼, and S. Du 杜石然. 1963. 中國古代數學簡史 (*Concise History of Mathematics in Ancient China*). Beijing: Zhongguo qinghua chubanshe.
Li, Yan, and S. Du. 1987. *Chinese Mathematics: A Concise History*. Trans. J. N. Crossley and A. W. C. Lun. Oxford: Clarendon Press.
Li Ye 李冶. 1248. *Ce yuan haijing* 測圓海鏡 (*Sea Mirror of the Measurements of the Circle*). Tongwenguan edition, 1876.
Martzloff, J. C. 1997. *A History of Chinese Mathematics*. Trans. S. S. Wilson. Berlin: Springer.
Qian, B. 錢寶琮. 1966. 宋元數學史論文集 (*Collected Papers on the History of Mathematics during Song and Yuan Dynasties*). Beijing: Science Press.
Qian, B. 錢寶琮. 1981. 中國數學史 (*History of Mathematics in China*). Beijing: Science Press.
Rashed, R. 1986. *Sharaf Al-Din Al-Tusi: Œuvres mathématiques*. 2 vols. Paris: Les Belles Lettres.
Shen, K. 沈康身. 1997. 九章算術導讀 (*Guide for the Reading of "The Nine Chapters"*). Hankou: Hubei jiaoyu chubanshe.
Shen, K., J. N. Crossley, and A. W.-C. Lun. 1999. *The Nine Chapters on the Mathematical Art: Companion and Commentary*. Oxford and Beijing: Oxford University Press and Science Press.
Te, G. 特古斯. 1990. "劉益及其佚著«議古根源»" ("Liu Yi and His Lost Book *Discussing the Source of the Ancient (Methods)*"). 數學史研究文集 (*Collected Research Papers on the History of Mathematics*) 1: 56–63.
Yan, D. 嚴敦傑. 1966. "宋楊輝算書考" ("Examination of Mathematical Books by Yang Hui of the Song Dynasty"). In 宋元數學史論文集 (*Collected Papers on the History of Mathematics during Song and Yuan Dynasties*), ed. B. Qian 錢寶琮, 149–165. Beijing: Science Press.
Yongle dadian 永樂大典 (*Grand Classic of the Yongle Period*). 1408. The part on mathematics of this encyclopedia ranged from chapter 16336 to 16357. Today only chapters 16343 and 16344 remain and are kept at Cambridge University Library. Reprint: Beijing 北京: zhonghua shuju 中華書局, 1960.
Yang Hui 楊輝. 1261. *Xiangjie jiuzhang suanfa* 詳解九章算法 (*Detailed Explanations of The Nine Chapters on Mathematical Methods*). Edition from the *Yijiatang congshu* 宜稼堂叢書, edited by Yu Songnian 郁松年, 1842.

BRUNO BELHOSTE is Professor of History of Science at the University of Paris 1. His main research interest is the development of science in France in the eighteenth and nineteenth centuries. He is the author of *Histoire de la science moderne de la Renaissance aux Lumières* (2016) and *Paris savant. Parcours et rencontres au temps des Lumières* (2011).

KARINE CHEMLA is Senior Researcher at the French National Center for Scientific Research (CNRS), in the laboratory SPHERE (CNRS and University Paris Diderot), and the European Research Council project SAW (Mathematical sciences in the ancient world) (https://sawerc.hypotheses.org). She focuses, from a historical and anthropology viewpoint, on the relationship between mathematics and the cultural contexts in which it is practiced. Chemla published, with Guo Shuchun, *Les neuf chapitres* (2004). She edited *The History of Mathematical Proof in Ancient Traditions* (2012); *Texts, Textual Acts and the History of Science* (with J. Virbel, 2015); and *The Oxford Handbook of Generality in Mathematics and the Sciences* (with R. Chorlay and D. Rabouin, 2016).

CAROLINE EHRHARDT is Maître de conférences (associate professor) in the History of Science at the Université Paris 8 Vincennes Saint-Denis (France). Her research investigates the social and cultural history of mathematics. She has worked on the history of algebra in the nineteenth century and on the history of French secondary and higher mathematics education during the modern and contemporary periods. Her publications include *Évariste Galois, la fabrication d'une icône mathématique* (2011) and *Itinéraire d'un texte mathématique: les réélaborations d'un mémoire d'Évariste Galois au 19e siècle* (2012).

FA-TI FAN is the author of *British Naturalists in Qing China: Science, Empire, and Cultural Encounter* (2003) and numerous articles on science in twentieth-century China and on the global history of science.

KENJI ITO is Associate Professor at SOKENDAI (the Graduate University for Advanced Studies), Hayama, Japan. The main area of his research is the history of physical sciences and technology in twentieth-century Japan. Topics of his publications include the history of physics in Japan, cultural images of robots and A-bombs, and amateur videogame culture in contemporary Japan. He is currently working on a biography of Nishina Yoshio and a book on the introduction of quantum mechanics into Japan.

EVELYN FOX KELLER is Professor Emerita of History and Philosophy of Science in the Program in Science, Technology and Society at Massachusetts Institute of Technology. She received her PhD in theoretical physics at Harvard University, worked for a number of years at the interface of physics and biology, and then turned to the study of gender and science, and more generally, to the history and philosophy of science. She is the author of numerous books, the recipient of many awards and honorary degrees, a member of the American Philosophical Society, the American Academy of Arts and Sciences, a MacArthur Fellow, and recipient of the Chaire Blaise Pascal in Paris.

GUILLAUME LACHENAL is Associate Professor in the Department of History of Science at the Université Paris Diderot. His research focuses on the history and anthropology of biomedicine in Africa, from the colonial times to the contemporary global health era. He has recently published *Le médicament qui devait sauver l'Afrique* (2014; English translation is forthcoming) and coedited the catalogue *Traces of the Future*, an archeology of medical science in Africa (2016).

DONALD MACKENZIE is Professor of Sociology at the University of Edinburgh. He is a sociologist and historian of science and technology. His current research is on the sociology of financial markets, in particular the development of automated high-frequency trading and of the electronic markets that make it possible, with a special focus on how trading algorithms predict the future. His books include *Inventing Accuracy: A Historical Sociology of Nuclear Missile Guidance* (1990); *An Engine, Not a Camera: How Financial Models Shape Markets* (2006); and *Material Markets: How Economic Agents Are Constructed* (2009).

MARY S. MORGAN is the Albert O. Hirschman Professor of History and Philosophy of Economics at the London School of Economics and holds a visiting fellowship at University of Pennsylvania. She is an elected Fellow of the British Academy, and an Overseas Fellow of the Royal Dutch Academy of Arts and Sciences. Her research interests cover questions about models, measurements, observation, experiments—the practical side of economics. Her two most recent books are *The World in the Model* (2012) and *How Well Do Facts Travel?* (2011). She is currently working on poverty measurement, the performativity of economics, and narrative explanation in the sciences.

NANCY J. NERSESSIAN is Regents' Professor (emerita), Georgia Institute of Technology. She currently is Research Associate in the Department of Psychology at Harvard University. Her research focuses on the creative research practices of scientists and engineers, especially how modeling practices lead to fundamentally new ways of understanding the world. This research has been funded by the National Science Foundation and National Endowment for the Humanities. She is a Fellow of American Association for the Advancement of Science and of the Cognitive Science Society as well as a Foreign Member of the Royal Netherlands Academy of Arts and Sciences. Her numerous publications include *Creating Scientific Concepts* (2008; Patrick Suppes Prize in Philosophy of Science, 2011) and *Science as Psychology: Sense-Making and Identity in Science Practice* (with L. Osbeck, K. Malone, and W. Newstetter, 2011; American Psychological Association William James Book Prize, 2012).

DAVID RABOUIN is Senior Research Fellow (CR1) at the French National Center for Scientific Research (CNRS) in the research group SPHERE (CNRS and Université Paris Diderot). His interest is in the history of philosophy and mathematics in early modern times, with a special focus on Descartes and Leibniz. He also works in contemporary French philosophy. He is the author of *Mathesis universalis: L'idée de « mathématique universelle » d'Aristote à Descartes* (2009). With Norma B. Goethe and Philip Beeley, he edited the collection titled *G.W. Leibniz, Interrelations between Mathematics and Philosophy* (2015).

HANS-JÖRG RHEINBERGER is Director Emeritus at the Max Planck Institute for the History of Science in Berlin and a molecular biologist and historian of science. His current research interests include the history and epistemology of experimentation, the history of the life sciences, and the relation between the sciences and the arts. Among his recent publications are *On Historicizing Epistemology* (2010); *An Epistemology of the Concrete* (2010); *A Cultural History of Heredity* (2012, with Staffan Müller-Wille); "Culture and Nature in the Prism of Knowledge" (*History of the Humanities* 1, no. 1, 2016).

CLAUDE ROSENTAL is Research Professor of Sociology at Centre National de la Recherche Scientifique, Director of the Center for the Study of Social Movements at the École des Hautes Études en Sciences Sociales, and head of the Science and Technology Group at Institut Marcel Mauss in Paris. His publications include works on the sociology of science, logic, and public demonstrations. He is the author of *Weaving Self-Evidence: A Sociology of Logic* (2008), *Les capitalistes de la science* (2007), and *La cognition au prisme des sciences sociales* (with B. Lahire, 2008).

KOEN VERMEIR is Senior Researcher at the CNRS laboratory SPHERE (University of Paris-Diderot). He is a Global Young Academy Fellow and co-directs the project Cultures of Baroque Spectacle. As a historian and philosopher specializing in early modern science, religion, and technology, he is now working on the long-term history of the concept of culture. Before coming to Paris, Vermeir has held positions in the United States, the United Kingdom, Belgium, Germany, and Switzerland. He is on the editorial boards of *Journal of Early Modern Studies*, *Society and Politics*, *Artefact*, *Studies in History and Philosophy of Science*, and *International Archives of the History of Ideas*.